ERROR ANALYSIS OF SOME MULTI-SYMPLECTIC SCHEME FOR KLEIN-GORDON-SCHRÖDINGER EQUATION

JIALIN HONG¹, SHANSHAN JIANG¹,², AND CHUN LI¹,²

Abstract. In this paper, we study the multi-symplectic method for the Klein-Gordon-Schrödinger equation. Besides conserving the multi-symplectic structure of the equation, the multi-symplectic method is also investigated for the conservation of charge, local energy, and total energy under corresponding discretization numerically. We deduce the transit law of the specific formal energy for the method. Based on these, we give the error estimations of the numerical solutions theoretically and present the results that the numerical solutions converge to the exact solutions for proper norms. In order to testify the superiority of the multi-symplectic method, it is compared with some non-multi-symplectic method. Numerical experiments show that the multi-symplectic method preserves the charge conservation law precisely, and conserves other conservation laws in better magnitude than the non-multi-symplectic method.

Keywords: Error analysis; conservation laws; multi-symplectic scheme; Klein-Gordon-Schrödinger equation.

1. Introduction

During the last few years, the multi-symplectic methods have been proposed and developed for some famous Hamiltonian partial differential equations, such as Dirac equations [4], wave equations [11], etc, for their perfectly preserving the geometry structure of equations.

In this paper, we pay our attention to the Klein-Gordon-Schrödinger equation, which describes a system of conserved scalar nucleons interacting with neutral scalar mesons. The Klein-Gordon-Schrödinger equation is one of important mathematical models in quantum physics, for it consists of one Klein-Gordon equation and one Schrödinger equation. Since Fukuda and Tsutsumi in [3] gave the first theoretical studies for Klein-Gordon-Schrödinger system, a great amount of work has been done on existence of solutions, stability [9] and asymptotic behavior [10]. But up to now, few numerical methods and simulation proposed for the Klein-Gordon-Schrödinger system as we know. In [12], Zhang presented a conservative difference scheme for a class of Klein-Gordon-Schrödinger equations, and present the convergence of the methods. It has only theoretical analysis there, but no numerical experiments. In Bao and Yang [1], the authors present spectral methods for Klein-Gordon-Schrödinger equation. Kong in [8] gives the multi-symplectic methods for Klein-Gordon-Schrödinger equation, and discusses the charge conservation law theoretically. In this paper, we investigate the...
multi-symplectic methods for Klein-Gordon-Schrödinger system, which is different from the angle of [1, 8, 12].

The organization of this paper is as follows. In the rest of this section, we introduce the preliminary knowledge about Klein-Gordon-Schrödinger system and some physical properties. In section 2, we present the theoretical analysis of the multi-symplectic integrators constructed by Runge-Kutta-type methods, and mention the preservation of the discrete multi-symplectic conservation law and the discrete charge conservation law under the mentioned discretization. Furthermore, we investigate the recursion reformulation of a specific formal energy for the multi-symplectic method we present. In section 3, we focus on developing the error estimations of the numerical solutions for Klein-Gordon-Schrödinger equation and prove that the numerical solutions converge to the theoretical solutions in some order. In section 4, we perform plentiful numerical experiments to testify the effectiveness of the multi-symplectic scheme listed in the previous section, and compare with some non-multi-symplectic scheme, and give a detailed comparison for the numerical results obtained by different schemes.

In the following we consider the standard Klein-Gordon-Schrödinger equation

\begin{align}
\left\{ \begin{array}{l}
\iota \partial_t \varphi + \frac{1}{2} \partial_{xx} \varphi + u \varphi = 0, \\
\partial_{tt} u - \partial_{xx} u + u - |\varphi|^2 = 0,
\end{array} \right.
\end{align}

where \( \varphi(x, t) \) denotes a complex scalar nucleon field, and \( u(x, t) \) denotes a real scalar meson field, respectively. Moreover, \( \iota = \sqrt{-1} \), and we denote the spatial and temporal direction by \((x, t)\), respectively, and \( x \in \mathbb{R}, t \geq 0 \). We supplement (1.1) by prescribing the initial-boundary value conditions for \( \varphi(x, t) \) and \( u(x, t) \) with:

\begin{align}
(1.2) \quad & \varphi|_{t=0} = \varphi_0(x), \quad u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \\
& \lim_{|x| \to \infty} \varphi = 0, \quad \lim_{|x| \to \infty} u = 0,
\end{align}

where, \( \varphi_0(x), u_0(x), \) and \( u_1(x) \) are known smooth functions.

If we set the complex valued function \( \varphi(x, t) = q(x, t) + \iota p(x, t) \), and introduce new variables \( g = \partial_x q, \ f = \partial_x p, \ v = \partial_t u, \ w = \partial_x u \), and state variable \( z = (q, p, g, f, u, v, w)^T \), the above equation (1.1) can be reformulated into the multi-symplectic Hamiltonian PDEs form

\begin{align}
(1.3) \quad & M \partial_t z + K \partial_x z = \nabla_z S(z),
\end{align}

where \( M \) and \( K \) are two skew-symmetric matrices (which can be singular),

\[ M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \iota^2 & 0 \\
0 & 0 & 0 & 0 & -\iota^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & -\iota^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\iota^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\iota^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.\]
and $S(z)$ is a real smooth function of the state variable $z$, which for system (1.1) is in the form

$$S(z) = \frac{1}{2} u(q^2 + p^2) + \frac{1}{4} (g^2 + f^2 - u^2 - v^2 + w^2).$$

It is well-known that one of the most interesting properties for (1.3) is that it possesses the multi-symplectic conservation law

$$(1.4) \quad \partial_t \omega + \partial_x \kappa = 0,$$

where

$$\omega = dz \wedge Mdz, \quad \kappa = dz \wedge Kdz,$$

which represents the geometric character on the phase space.

It can be easily seen that (1.4) is a natural extension of conservation of symplecticity from Hamiltonian ODEs to Hamiltonian PDEs, which indicates that symplecticity can vary over the spatial domain and from time to time, and this variation is not arbitrary as the changes in time are exactly compensated by changes in space (see [7]).

Specially, (1.4) can be rewritten into the equivalent form

$$(1.5) \quad \partial_t [2dq \wedge dp + du \wedge dv] + \partial_x [dg \wedge dq + df \wedge dp + dw \wedge dv] = 0$$

for the Klein-Gordon-Schrödinger equation (1.1).

A more spiriting fact is that there exist several local conservation laws along with the multi-symplecticity of Hamiltonian PDEs, which are local energy conservation law

$$(1.6) \quad \partial_t E(z) + \partial_x F(z) = 0,$$

where

$$E(z) = S(z) - \frac{1}{2} z^T K \partial_x z, \quad F(z) = \frac{1}{2} z^T K \partial_t z,$$

represent energy density and energy flux, respectively, and local momentum conservation law

$$(1.7) \quad \partial_t I(z) + \partial_x G(z) = 0,$$

where

$$G(z) = S(z) - \frac{1}{2} z^T M \partial_t z, \quad I(z) = \frac{1}{2} z^T M \partial_x z.$$
Integrating the local energy conservation law (1.6) over the spatial domain, and recalling the boundary conditions in (1.2), we derive the conservation of total energy (1.8)
\[
\frac{d}{dt} E(z) := \frac{d}{dt} \int_{\mathbb{R}} E(z) dx = 0,
\]
where \( E(z) = \int_{\mathbb{R}} \left( u^2 + v^2 + w^2 + f^2 + g^2 - 2u(q^2 + p^2) \right) dx \). This is one of important global conservative quantities for Klein-Gordon-Schrödinger equation (1.1). Furthermore, by a straight calculation, we have the following proposition. It gives one intrinsic conservative quantity for Kilein-Gordon-Schrödinger equation itself.

**Proposition 1.1.** The solution \( \varphi \) of the initial-boundary value problem (1.1)-(1.2) satisfies the charge conservation law (1.9)
\[
\int_{\mathbb{R}} |\varphi(x, t)|^2 dx = \int_{\mathbb{R}} |\varphi_0(x)|^2 dx = C,
\]
where, \( C \) is a constant.

In the subsequent section, we will turn attention to applying some multi-symplectic scheme to the Klein-Gordon-Schrödinger equation and investigate some discrete conservative quantities under the numerical discretization.

### 2. The special multi-symplectic scheme and its conservative laws

In this section, we focus on investigating the central box scheme, which is multi-symplectic. For convenience, the following notations are used:
\[
\delta t V^{k+\frac{1}{2}}_j = \frac{1}{\tau} (V^{k+1}_j - V^k_j), \quad \delta t^2 V^k_j = \frac{1}{\tau^2} (V^{k+1}_j - 2V^k_j + V^{k-1}_j), \quad \delta x^2 V^k_j = \frac{1}{h^2} (V^{k+1}_j - 2V^k_j + V^{k-1}_j),
\]
and
\[
(U, V) = h \sum_j U_j V_j, \quad \|V^k\|^2 := h \sum_j |V^k_j|^2, \quad \|V^k\|_2 := h \sum_j |V^{k-\frac{1}{2}}_j|^2.
\]

Here, we choose \( h \) as the spatial mesh grid-size, and \( \tau \) as the time stepsize, respectively. And introduce a uniform grid \((x_j, t_k) \in \mathbb{R}^2\), then the approximation of the value of function \( V(x, t) \) at the mesh grid \((x_j, t_k)\) is denoted by \( V^k_j \).

When we apply the central box scheme to the equation (1.1), then have the following numerical scheme
\[
\begin{align*}
        &u^{k+\frac{1}{2}}_{j+\frac{1}{2}} - u^{k\frac{1}{2}}_{j+\frac{1}{2}} = \tau v^{k\frac{1}{2}}_{j+\frac{1}{2}}, \\
        &u^{k\frac{1}{2}}_{j+1} - u^{k\frac{1}{2}}_{j} = hu^{k\frac{1}{2}}_{j+\frac{1}{2}}, \\
        &q^{k\frac{1}{2}}_{j+1} - q^{k\frac{1}{2}}_{j} = hg^{k\frac{1}{2}}_{j+\frac{1}{2}}, \\
        &p^{k\frac{1}{2}}_{j+1} - p^{k\frac{1}{2}}_{j} = hf^{k\frac{1}{2}}_{j+\frac{1}{2}}, \\
\end{align*}
\]
(2.1)
The implicit midpoint discretization (2.1)-(2.2) is multi-symplectic, numerically. This version of the geometric property on the phase space. However, this can not be verified theoretically (see [2, 4, 5] and references therein):

\[
\begin{align*}
\frac{q_{j+\frac{1}{2}}^{k+1} - q_{j+\frac{1}{2}}^{k}}{\tau} &+ \frac{1}{2} \frac{f_{j+\frac{1}{2}}^{k+1} - f_{j+\frac{1}{2}}^{k}}{h} + u_{j+\frac{1}{2}}^{k+\frac{1}{2}} p_{j+\frac{1}{2}}^{k+\frac{1}{2}} = 0, \\
\frac{p_{j+\frac{1}{2}}^{k+1} - p_{j+\frac{1}{2}}^{k}}{\tau} &- \frac{1}{2} \frac{g_{j+\frac{1}{2}}^{k+\frac{1}{2}} - g_{j}^{k}}{h} - u_{j+\frac{1}{2}}^{k+\frac{1}{2}} q_{j+\frac{1}{2}}^{k+\frac{1}{2}} = 0,
\end{align*}
\]

(2.2)

\[
\begin{align*}
\frac{v_{j+\frac{1}{2}}^{k+1} - v_{j+\frac{1}{2}}^{k}}{\tau} &- \frac{w_{j+1}^{k+\frac{1}{2}} - w_{j}^{k+\frac{1}{2}}}{h} + u_{j+\frac{1}{2}}^{k+\frac{1}{2}} - ((p_{j+\frac{1}{2}}^{k+\frac{1}{2}})^2 + (g_{j+\frac{1}{2}}^{k+\frac{1}{2}})^2) = 0,
\end{align*}
\]

in which, \(u_{j+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2}(u_{j}^{k} + u_{j+1}^{k}), u_{j}^{k+\frac{1}{2}} = \frac{1}{2}(u_{j}^{k+1} + u_{j}^{k}), v_{j}^{k+\frac{1}{2}} = \frac{1}{2}(v_{j}^{k+1} + v_{j}^{k}), q_{j}^{k+\frac{1}{2}} = \frac{1}{2}(q_{j}^{k+1} + q_{j}^{k}), p_{j}^{k+\frac{1}{2}} = \frac{1}{2}(p_{j}^{k+1} + p_{j}^{k}), f_{j}^{k+\frac{1}{2}} = \frac{1}{2}(f_{j}^{k+1} + f_{j+1}^{k}), g_{j}^{k+\frac{1}{2}} = \frac{1}{2}(g_{j}^{k} + g_{j+1}^{k}).\)

Then we give the following expression

\[
E_{j+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{E_{j+\frac{1}{2}}^{k+1} - E_{j+\frac{1}{2}}^{k}}{\tau} + \frac{E_{j+1}^{k+\frac{1}{2}} - E_{j}^{k+\frac{1}{2}}}{h}
\]

(2.3)

to denote the residual of local energy conservation law, though the multi-symplectic scheme can not preserve the corresponding discretization of local energy conservation law (1.6) exactly. In the numerical experiments, we will calculate such residual numerically, for it is an important sign for the local property of the multi-symplectic methods.

Similarly, we denote the discrete total energy (1.8) by

\[
E^{k+1} = h \sum_{j} E_{j+\frac{1}{2}}^{k+1}
\]

(2.4)

for use in the follows.

Using the similar straight calculations in reference [2], we can obtain the following discrete multi-symplectic conservation law

\[
\begin{align*}
\tau(dg_{j+\frac{1}{2}}^{k+\frac{1}{2}} &\wedge dq_{j+\frac{1}{2}}^{k+\frac{1}{2}} + df_{j+\frac{1}{2}}^{k+\frac{1}{2}} \wedge dp_{j+\frac{1}{2}}^{k+\frac{1}{2}} + dw_{j+\frac{1}{2}}^{k+\frac{1}{2}} \wedge dv_{j+\frac{1}{2}}^{k+\frac{1}{2}}) \\
- \tau(dg_{j+\frac{1}{2}}^{k+\frac{1}{2}} &\wedge dq_{j+\frac{1}{2}}^{k+\frac{1}{2}} + df_{j+\frac{1}{2}}^{k+\frac{1}{2}} \wedge dp_{j+\frac{1}{2}}^{k+\frac{1}{2}} + dw_{j+\frac{1}{2}}^{k+\frac{1}{2}} \wedge dv_{j+\frac{1}{2}}^{k+\frac{1}{2}}) \\
+ h(2dq_{j+\frac{1}{2}}^{k+1} &\wedge dp_{j+\frac{1}{2}}^{k+1} + dq_{j+\frac{1}{2}}^{k+1} \wedge dv_{j+\frac{1}{2}}^{k+1} + dq_{j+\frac{1}{2}}^{k+1} - h(2dq_{j+\frac{1}{2}}^{k+1} &\wedge dp_{j+\frac{1}{2}}^{k+1} + dq_{j+\frac{1}{2}}^{k+1} \wedge dv_{j+\frac{1}{2}}^{k+1})
\end{align*}
\]

(2.5)

which is the corresponding discretization version of (1.5). Then we have the following theorem (see [2, 4, 5] and references therein):

**Theorem 2.1.** The implicit midpoint discretization (2.1)-(2.2) is multi-symplectic, and it satisfies the discrete multi-symplectic conservation law (2.5).

It is well-known that the discrete multi-symplectic conservation law is the discrete versions of the geometric property on the phase space. However, this can not be verified numerically.
If the introduced variables $v, w, f, g$ are eliminated, the multi-symplectic scheme (2.1)-(2.2) can be reformulated as

\[
\frac{1}{\tau} (q_j^{k+\frac{1}{2}} - q_j^{k+\frac{1}{2}}) + \frac{1}{\tau} (q_j^{k+1} - q_j^{k-1}) + \frac{1}{h_2} (p_j^{k+\frac{1}{2}} - 2p_j^{k+\frac{1}{2}} + p_j^{k-\frac{1}{2}})
\]

(2.6)

\[
\frac{1}{\tau} (p_j^{k+\frac{1}{2}} - p_j^{k+\frac{1}{2}}) + \frac{1}{\tau} (p_j^{k+1} - p_j^{k-1}) + \frac{1}{h_2} (q_j^{k+\frac{1}{2}} - 2q_j^{k+\frac{1}{2}} + q_j^{k-\frac{1}{2}})
\]

(2.7)

Then we derive the following numerical scheme

\[
\frac{2}{\tau^2} (u_j^{k+1} - 2u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}}) - u_j^{k+1} - u_j^{k-1} - 2u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}} - \frac{2}{\tau^2} (u_j^{k+1} - 2u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}})
\]

(2.8)

Combining (2.6) with (2.7), we obtain

\[
\frac{1}{\tau} (\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}}) + \frac{1}{\tau} (\varphi_j^{k+1} - \varphi_j^{k-1}) + \frac{1}{h_2} (\varphi_j^{k+\frac{1}{2}} - 2\varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}})
\]

(2.9)

\[
\frac{1}{\tau} (\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}}) + \frac{1}{\tau} (\varphi_j^{k+1} - \varphi_j^{k-1}) + \frac{1}{h_2} (\varphi_j^{k+\frac{1}{2}} - 2\varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}})
\]

Then we derive the following numerical scheme

\[
i(\delta_t \varphi_j^{k+\frac{1}{2}} + \delta_t \varphi_j^{k-\frac{1}{2}}) + \delta_x^2 \varphi_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} = 0
\]

(2.10)

\[
2(\delta_x^2 u_j^{k+\frac{1}{2}} + \delta_x^2 u_j^{k-\frac{1}{2}}) - 2(\delta_x^2 u_j^{k+\frac{1}{2}} + \delta_x^2 u_j^{k-\frac{1}{2}}) + (u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}} + u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}})
\]

(2.11)

by using the notations defined previously.

Next, we turn to the discrete charge conservation law firstly.

**Theorem 2.2.** The multi-symplectic scheme (2.10)-(2.11) possesses the discrete charge conservation law, i.e.

\[
\|\varphi^k\|_2^2 = \text{Constant.}
\]

**Proof.** We first multiply both sides of (2.10) by $2\varphi_j^{k+\frac{1}{2}}$, i.e. $\varphi_j^{k+1} + \varphi_j^{k-1}$, where $\varphi$ denotes the complex conjugate of $\varphi$. It follows from the first term that

\[
\frac{1}{2\tau} (\varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} + \varphi_j^{k-\frac{1}{2}})
\]

(2.13)
The second term becomes

\begin{equation}
(2.14) \quad \frac{2}{h^2} (\varphi_{j+1}^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} + \varphi_{j-1}^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}}).
\end{equation}

Similarly, the rest of left side yields

\begin{equation}
(2.15) \quad u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+1}^{k+\frac{1}{2}} + u_{j-\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j-1}^{k+\frac{1}{2}} + \frac{k}{j+\frac{1}{2}} \varphi_{j}^{k+\frac{1}{2}} - \frac{k}{j-\frac{1}{2}} \varphi_{j}^{k+\frac{1}{2}} + u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j}^{k+\frac{1}{2}} + u_{j-\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j}^{k+\frac{1}{2}} - \frac{k}{j+\frac{1}{2}} \varphi_{j}^{k+\frac{1}{2}} \varphi_{j}^{k+\frac{1}{2}}.
\end{equation}

\[\text{summing over all spatial grid points } j, \text{ and taking the imaginary part, we can find that (2.14) and (2.15) both become real functions, and it follows from (2.13) that}\]

\[\frac{i}{2 \tau} h \sum_j [(\varphi_{j+1}^{k+1} \varphi_j^{k+1} + 2 \varphi_j^{k+1} \varphi_{j+1}^{k+1} + \varphi_{j-1}^{k+1} \varphi_j^{k+1}) - (\varphi_{j+1}^{k} \varphi_j^{k} + 2 \varphi_j^{k} \varphi_{j+1}^{k} + \varphi_{j-1}^{k} \varphi_j^{k})]\]

\[\text{with the zero boundary conditions.}\]

\[\text{From the analysis above, we can further deduce}\]

\begin{equation}
(2.16) \quad \|\varphi^{k+1}\|_2^2 = h \sum_j \varphi_{j+\frac{1}{2}}^{k+1} \varphi_{j+\frac{1}{2}}^{k+1} = h \sum_j \varphi_{j+\frac{1}{2}}^{k} \varphi_{j+\frac{1}{2}}^{k} = \|\varphi^k\|_2^2 = \cdots = \|\varphi^0\|_2^2.
\end{equation}

This completes the proof. \(\Box\)

\[\text{It can be seen evidently that Theorem 2.2 is consistent with the charge conservation law (1.9). And it means that the charge conservation law of KGS equation can be preserved by our multi-symplectic scheme exactly. In general, multi-symplectic Runge-Kutta type methods have the stability in the sense of the charge conservation law for multi-symplectic Hamiltonian PDEs, which can be referred to [4]. As a quadratic invariant the charge conservation law plays a very important part in quantum physics, however, it can not be preserved by symplectic integrators in the case of Hamiltonian ODEs, and this is just one of interesting things to introduce multi-symplectic methods.}\]

\[\text{The next result concerns the error estimation of the discrete total energy, and for the reason of the nonlinear terms in equation(1.1), the multi-symplectic method can not preserve the discrete total energy conservation law exactly.}\]

\[\text{Lemma 2.3. Under the zero boundary conditions, the multi-symplectic scheme (2.10)-(2.11) satisfies the following two implicit discrete global conservation laws, i.e.}\]

\[h \sum_j (\varphi_{j+1}^{k+\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}}) (\varphi_{j+1}^{k+\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}}) - (\varphi_{j+1}^{k-\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}}) (\varphi_{j+1}^{k-\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}})]

\[2h \sum_j (u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} - u_{j-\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j-\frac{1}{2}}^{k-\frac{1}{2}})\]

\[= 2h \sum_j \Re(u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} - u_{j-\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j-\frac{1}{2}}^{k-\frac{1}{2}}).\]
Here, \( \Re \) stands for ‘real’ part.

\[
\begin{align*}
&h \sum_j \left[ \frac{(u_{j+\frac{1}{2}}^k - u_{j}^k)^2}{\tau} - \frac{(u_{j+\frac{1}{2}}^{k-1} - u_{j}^{k-1})^2}{\tau} \right] + h \sum_j \left[ (u_{j+\frac{1}{2}}^{k+\frac{1}{2}})^2 - (u_{j+\frac{1}{2}}^{k-\frac{1}{2}})^2 \right] \\
&+ h \sum_j \left[ \frac{(u_{j+1} - u_j)^2}{h} - \frac{(u_{j+1} - u_j - \frac{k}{2})^2}{h} \right] \\
&= h \sum_j (\varphi_{j+\frac{1}{2}}^k \varphi_{j+\frac{1}{2}}^k + \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}})(u_{j+\frac{1}{2}}^k - u_{j+\frac{1}{2}}^{k-\frac{1}{2}}).
\end{align*}
\]

**Proof.** We can get from (2.10) that

\[
i(\delta_t \varphi_j^{k+\frac{1}{2}} + \delta_t \varphi_j^{k+\frac{1}{2}} + \delta_t \varphi_j^{k-\frac{1}{2}} + \delta_t \varphi_j^{k-\frac{1}{2}}) + (\delta^2 x \varphi_j^{k+\frac{1}{2}} + \delta^2 x \varphi_j^{k-\frac{1}{2}}) \\
+ u_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} + u_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} + u_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} = 0.
\]

We multiply the above equation by \( \varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \), sum over all spatial grid points \( j \), and take the real part. With zero boundary conditions, the first term becomes imaginary functions. It follows from the second term that

\[
h \sum_j \Re((\delta^2 x \varphi_j^{k+\frac{1}{2}} + \delta^2 x \varphi_j^{k-\frac{1}{2}})(\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}}))
\]

\[
= \frac{1}{h^2} \sum_j h \Re((\varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} + 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} + 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}})
\]

\[
= \frac{1}{h^2} \sum_j h \Re((\varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} + 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} + 2 \varphi_j^{k+\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}})
\]

\[
= h \sum_j \left( \frac{k+\frac{1}{2}}{h} \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - \frac{k-\frac{1}{2}}{h} \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} \right).
\]

Obtained from the rest terms in (2.10) that

\[
h \sum_j \Re((u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} + u_{j+\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}} + u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} + u_{j+\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}})(\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}})
\]

\[
= 2h \sum_j \Re((u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} + u_{j+\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}})(\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}}))
\]

\[
= 2h \sum_{j=0}^{J-1} \Re((u_{j+\frac{1}{2}}^{k+\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} + u_{j+\frac{1}{2}}^{k-\frac{1}{2}} \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}})(\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}})).
\]
Combining (2.19) with (2.20), we finally get

\[
\begin{align*}
&h \sum_{j=0}^{J-1} \left[ \left( \frac{\varphi_{j+1/2} + \varphi_{j-1/2}}{2h} \right) \left( \frac{\varphi_{j+1/2} - \varphi_{j-1/2}}{2h} \right) - \left( \frac{\varphi_{j+1} - \varphi_{j}}{h} \right) \left( \frac{\varphi_{j+1} - \varphi_{j}}{h} \right) \right] \\
&- 2h \sum_{j=0}^{J-1} \left( \frac{\varphi_{j+1/2} + \varphi_{j-1/2} - \varphi_{j+1} + \varphi_{j}}{2h} \left( \frac{\varphi_{j+1} - \varphi_{j}}{h} \right) \right) \\
&= 2h \sum_{j=0}^{J-1} \mathcal{R} \left( \frac{\varphi_{j+1} - \varphi_{j}}{2h} \frac{\varphi_{j+1} - \varphi_{j}}{2h} \right)
\end{align*}
\]

Similarly, we multiply (2.11) by \( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \), sum over all spatial grid points \( j \). By making use of zero boundary conditions, it follows from the first term that

\[
\begin{align*}
&h \sum_{j} 2 \left( \delta_x^2 u_{j+\frac{1}{2}} + \delta_x^2 u_{j-\frac{1}{2}} \right) \left( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \right) \\
&= \frac{h}{2\tau^2} \sum_{j} \left( \left( u_{j+1} + u_{j+1}^{k+1} + u_{j+1} - u_{j+1}^{k-1} \right) - \left( u_{j+1} + u_{j+1}^{k+1} - u_{j+1}^{k-1} \right)^2 \right) \\
&= 2h \sum_{j} \left( \left( u_{j+1}^{k+1} - u_{j+1}^{k-1} \right) \left( u_{j+1}^{k+1} - u_{j+1}^{k-1} \right) \right)
\end{align*}
\]

The second term becomes

\[
\begin{align*}
-2h \sum_{j} \left( \delta_x^2 u_{j+\frac{1}{2}} + \delta_x^2 u_{j-\frac{1}{2}} \right) \left( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \right) = 2h \sum_{j} \left( \frac{u_j^{k+1} - u_j^{k-1}}{h} \right)^2 - \left( \frac{u_j^{k+1} - u_j^{k-1}}{h} \right)^2.
\end{align*}
\]

The third term yields

\[
\begin{align*}
&h \sum_{j} \left( u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}} + u_j^{k+\frac{1}{2}} + u_j^{k-\frac{1}{2}} \right) \left( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \right) = 2h \sum_{j} \left( u_j^{k+1} - u_j^{k-1} \right)^2.
\end{align*}
\]

The rest terms read

\[
\begin{align*}
&- h \sum_{j} \left( \left| \varphi_{j+\frac{1}{2}} \right|^2 + \left| \varphi_{j-\frac{1}{2}} \right|^2 + \left| \varphi_{j+\frac{1}{2}} \right|^2 + \left| \varphi_{j-\frac{1}{2}} \right|^2 \right) \left( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \right) \\
&= -2h \sum_{j} \left( \varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}} - \varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}} \right) \left( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \right).
\end{align*}
\]
Combining (2.21), (2.22), (2.23) and (2.24), we get
\[
\begin{align*}
h \sum_j & \left( \frac{\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k+\frac{1}{2}}}{\tau} \right)^2 - \left( \frac{u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}}}{\tau} \right)^2 \right) + h \sum_j \left( \frac{u_j^{k+1} - u_j^{k-1}}{\tau} \right)^2 \\
+ h & \sum_j \left( \frac{u_{j+1}^{k+\frac{1}{2}} - u_{j}^{k+\frac{1}{2}}}{h} \right)^2 - \left( \frac{u_{j+1}^{k-\frac{1}{2}} - u_{j}^{k-\frac{1}{2}}}{h} \right)^2 \\
= h & \sum_j \left( \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} \right) (u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}}).
\end{align*}
\]

This completes the proof. \(\square\)

According to Lemma 2.3, if we set the corresponding discrete total energy of the multi-symplectic scheme (2.10)-(2.11) at time \(t_{k+\frac{1}{2}}\) as

\[
(2.25)
\]

\[
\mathcal{E}_j^{k+\frac{1}{2}} = h \sum_{j=0}^{J-1} \left( \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} \right) (u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}})
+ \left( \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} \right),
\]

we have the following result about the transit of the discrete total energy in temporal direction. Furthermore, (2.25) can be considered as the discrete version of the total energy conservation law (1.8) at time \(t_{k+\frac{1}{2}}\), when applying the multi-symplectic scheme (2.10)-(2.11) to system (1.1).

**Theorem 2.4.** For the multi-symplectic scheme (2.10)-(2.11), the transit of the discrete total energy in the temporal direction obeys the following law

\[
(2.26)
\]

\[
\mathcal{E}_j^{k+\frac{1}{2}} - \mathcal{E}_j^{k-\frac{1}{2}} = h \sum_{j=0}^{J-1} (u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}}) (\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}})^2.
\]

**Proof.** Due to the implicit discrete global conservation laws (2.17)-(2.18) in Lemma 2.3, we obtain

\[
\begin{align*}
\mathcal{E}_j^{k+\frac{1}{2}} - \mathcal{E}_j^{k-\frac{1}{2}} = h & \sum_j \left( \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} \right) (u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}})
+ \left( \varphi_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}} \varphi_j^{k-\frac{1}{2}} \right) (u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}})
= \sum_j \left( u_j^{k+\frac{1}{2}} - u_j^{k-\frac{1}{2}} \right) (\varphi_j^{k+\frac{1}{2}} - \varphi_j^{k-\frac{1}{2}})^2.
\end{align*}
\]

This completes the proof. \(\square\)

The discrete total energy at time \(t_{k+\frac{1}{2}}\) is the specific property for the multi-symplectic scheme we present, for its special geometry structure.
3. Error estimations for the multi-symplectic scheme

In this section, we investigate the properties of the multi-symplectic scheme (2.10)-(2.11) further, and calculate its convergence, which is the most important sign of the effectiveness of the numerical methods. For the convenience of readers, We set truncation errors of scheme as

\[ r_j^{k+\frac{1}{2}} = i(\delta_t \varphi(x_j^{+\frac{1}{2}}, t_{k+\frac{1}{2}}) + \delta_x \varphi(x_j^{-\frac{1}{2}}, t_{k+\frac{1}{2}})) + \delta_x^2 \varphi(x_j, t_{k+\frac{1}{2}}) + u(x_j^{+\frac{1}{2}}, t_{k+\frac{1}{2}}) \varphi(x_j^{-\frac{1}{2}}, t_{k+\frac{1}{2}}), \]

(3.1)

\[ \sigma_j^k = 2(\delta_t^2 u(x_j^{+\frac{1}{2}}, t_k) + \delta_t^2 u(x_j^{-\frac{1}{2}}, t_k)) - 2(\delta_t^2 u(x_j, t_{k+\frac{1}{2}}) + \delta_t^2 u(x_j, t_{k-\frac{1}{2}})) + u(x_j^{+\frac{1}{2}}, t_{k+\frac{1}{2}}) + u(x_j^{-\frac{1}{2}}, t_{k-\frac{1}{2}}) \]

(3.2)

Let global errors of solutions be denoted by \( e_j^k = \varphi(x_j, t_k) - \varphi_j^k, \quad \eta_j^k = u(x_j, t_k) - u_j^k. \)

It can be verified by Taylor expansion that the truncation errors of the multi-symplectic scheme (2.10)-(2.11) are \( O(\tau^2 + h^2), \) i.e.

\[ r_j^{k+\frac{1}{2}} = O(\tau^2 + h^2), \quad \sigma_j^k = O(\tau^2 + h^2). \]

(3.3)

According to the relationship of recursion, we can obtain the following expression

\[ r_j^{k+\frac{1}{2}} = \frac{1}{2}(r_j^{k+\frac{1}{2}} + r_j^{k-\frac{1}{2}}) + O(\tau^2 + h^2), \quad \sigma_j^k = \frac{1}{2}(\sigma_j^{k+\frac{1}{2}} + \sigma_j^{k-\frac{1}{2}}) + O(\tau^2 + h^2). \]

(3.4)

Then we have the following global error estimations for the multi-symplectic schemes (2.10)-(2.11).

**Theorem 3.1.** For the the multi-symplectic scheme (2.10)-(2.11), the following estimates hold:

\[
\{ \| \delta \eta_j^{N+\frac{1}{2}} \|^2_2, \| \delta_x \eta_j^{N+\frac{1}{2}} \|^2_2, \| \eta_j^{N+\frac{1}{2}} \|^2_2, \| e_j^{N+\frac{1}{2}} \|^2_2 \} \leq \{(1 + C\tau)(\| \delta \eta_j^{N+\frac{1}{2}} \|^2_2 + \| \delta_x \eta_j^{N+\frac{1}{2}} \|^2_2 + \| \eta_j^{N+\frac{1}{2}} \|^2_2 + \| e_j^{N+\frac{1}{2}} \|^2_2) + \tau \sum_{k=1}^{N} (r_j^{k+\frac{1}{2}})^2_2 + r_j^{k-\frac{1}{2}})^2_2 + \frac{1}{4} \| \sigma_j^k \|^2_2 + O(\tau^2 + h^2) \}\} \exp^{AN\tau},
\]

(3.5)

where, \( \tau \) is sufficiently small, such that \( \tau \leq \frac{1}{4C}. \)

**Proof.** Firstly, it follows from subtracting (2.10) from (3.1) that

\[ r_j^{k+\frac{1}{2}} = i(\delta_t e_j^{k+\frac{1}{2}} + \delta_x e_j^{k+\frac{1}{2}}) + \delta_x^2 e_j^{k+\frac{1}{2}} + u(x_j^{+\frac{1}{2}}, t_{k+\frac{1}{2}}) \varphi(x_j^{-\frac{1}{2}}, t_{k+\frac{1}{2}}) \]

(3.6)

\[ + u(x_j^{-\frac{1}{2}}, t_{k+\frac{1}{2}}) \varphi(x_j^{+\frac{1}{2}}, t_{k+\frac{1}{2}}) - \frac{1}{4} \sigma_j^k \varphi_j^{k+\frac{1}{2}} - \frac{1}{4} \sigma_j^k \varphi_j^{k-\frac{1}{2}}. \]
Multiplying (3.6) by $e^{k+\frac{1}{2}}$, summing over all the spatial grid points $j$ and taking the imaginary part, we obtain

$$\frac{1}{\tau}(\|e^{k+1}\|_2^2 - \|e^k\|_2^2) = \Im h \sum_j (r_j^{k+\frac{1}{2}} e^{k+\frac{1}{2}}) - 2\Im h \sum_j (u(x_{j+\frac{1}{2}}, t_{k+\frac{1}{2}}) \varphi(x_{j+\frac{1}{2}}, t_{k+\frac{1}{2}}) - u_{j+\frac{1}{2}}^k \varphi_{j+\frac{1}{2}}^k e^{k+\frac{1}{2}}).$$

Here, $\Im$ stands for ‘imaginary’ part.

Using $\sum_j r_j^{k+\frac{1}{2}} e^{k+\frac{1}{2}} = \sum_j (r_j^{k+\frac{1}{2}} + \mathcal{O}(\tau^2 + h^2)) e^{k+\frac{1}{2}}$, we have

$$\frac{1}{\tau}(\|e^{k+1}\|_2^2 - \|e^k\|_2^2) = \Im h \sum_j (r_j^{k+\frac{1}{2}} + \mathcal{O}(\tau^2 + h^2)) e^{k+\frac{1}{2}} - 2\Im h \sum_j (\eta_j^{k+\frac{1}{2}} \varphi_j^{k+\frac{1}{2}}) e^{k+\frac{1}{2}}.$$  

(3.7)

It follows from Theorem 2.2, there exists a constant

$$C_1 = \max(||\varphi_{j+\frac{1}{2}}||), \quad j = 0, \ldots; \quad k = 0, \ldots, N$$

such that

$$\|e^{k+1}\|_2^2 - \|e^k\|_2^2 \leq \tau [\|r^{k+\frac{1}{2}}\|_2^2 + \frac{1}{2}\|e^{k+\frac{1}{2}}\|_2^2 + C_1 ||\eta^{k+\frac{1}{2}}\|_2^2 + C_1 ||\varphi^{k+\frac{1}{2}}\|_2^2 + \mathcal{O}(\tau^2 + h^2)]$$

$$\leq \tau [\|r^{k+\frac{1}{2}}\|_2^2 + C_1 ||\eta^{k+\frac{1}{2}}\|_2^2 + \frac{1}{2}(C_1 + \frac{1}{2})(\|e^{k+1}\|_2^2 + \|e^k\|_2^2) + \mathcal{O}(\tau^2 + h^2)].$$

On the other hand, substituting (2.11) into (3.2), we have

$$\sigma_j^k = 2(\delta_x^2 \eta_j^{k+\frac{1}{2}} + \delta_x^2 \eta_j^{k-\frac{1}{2}}) - 2(\delta_x^2 \eta_j^{k+\frac{1}{2}} + \delta_x^2 \eta_j^{k-\frac{1}{2}}) + \eta_j^{k+\frac{1}{2}} + \eta_j^{k-\frac{1}{2}} + \eta_j^{k+\frac{1}{2}} + \eta_j^{k-\frac{1}{2}}$$

$$- |\varphi(x_{j+\frac{1}{2}}, t_{k+\frac{1}{2}})|^2 - |\varphi(x_{j-\frac{1}{2}}, t_{k+\frac{1}{2}})|^2 - |\varphi(x_{j-\frac{1}{2}}, t_{k-\frac{1}{2}})|^2 - |\varphi(x_{j-\frac{1}{2}}, t_{k-\frac{1}{2}})|^2$$

$$+ |\varphi_j^{k+\frac{1}{2}}|^2 + |\varphi_j^{k-\frac{1}{2}}|^2 + |\varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}}|^2 + |\varphi_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2$$

(3.9)

Multiplying (3.9) by $\frac{1}{2}(\eta_j^{k+\frac{1}{2}} - \eta_j^{k-\frac{1}{2}})$, and summing over $j$, one has

$$\frac{2}{\tau}(\|\delta_x\eta^{k+\frac{1}{2}}\|_2^2 - \|\delta_x\eta^{k-\frac{1}{2}}\|_2^2 + \|\delta_x\eta^{k+\frac{1}{2}}\|_2^2 - \|\delta_x\eta^{k-\frac{1}{2}}\|_2^2 + \|\eta^{k+\frac{1}{2}}\|_2^2 - \|\eta^{k-\frac{1}{2}}\|_2^2)$$

$$= \frac{1}{2} h \sum_j \sigma_j^k (\delta \eta_j^{k+\frac{1}{2}} + \delta \eta_j^{k-\frac{1}{2}})$$

$$+ h \sum_j (|\varphi(x_{j+\frac{1}{2}}, t_{k+\frac{1}{2}})|^2 + |\varphi(x_{j+\frac{1}{2}}, t_{k-\frac{1}{2}})|^2 - |\varphi_{j+\frac{1}{2}}^{k+\frac{1}{2}}|^2 - |\varphi_{j+\frac{1}{2}}^{k-\frac{1}{2}}|^2)(\delta \eta_j^{k+\frac{1}{2}} + \delta \eta_j^{k-\frac{1}{2}}).$$
From $\sum_{j} \sigma_{j}^{k}(\delta_{j}^{k+\frac{1}{2}} + \delta_{j}^{k-\frac{1}{2}}) = \sum_{j}(\sigma_{j}^{k+\frac{1}{2}} + O(\tau^{2} + h^{2}))(\delta_{j}^{k+\frac{1}{2}} + \delta_{j}^{k-\frac{1}{2}})$, we obtain

$$\left(\frac{1}{\tau}(\|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2}) + \|\delta_{j}\|_{2}^{2} + \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}\|_{2}^{2} - \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}\|_{2}^{2})ight)$$

$$\leq \frac{1}{4}h \sum_{j}(\sigma_{j}^{k+\frac{1}{2}} + O(\tau^{2} + h^{2}))(\delta_{j}^{k+\frac{1}{2}} + \delta_{j}^{k-\frac{1}{2}}) + \frac{1}{2}h \sum_{j}(\varphi_{j}^{k+\frac{1}{2}} + \varphi_{j}^{k-\frac{1}{2}})\varphi(x_{j} + \frac{1}{2}, t_{k+\frac{1}{2}}) + \varphi_{j}^{k+\frac{1}{2}}e_{j}^{k+\frac{1}{2}} + \varphi_{j}^{k-\frac{1}{2}}e_{j}^{k-\frac{1}{2}} + \varphi_{j}^{k+\frac{1}{2}}e_{j}^{k+\frac{1}{2}} + \varphi_{j}^{k-\frac{1}{2}}e_{j}^{k-\frac{1}{2}}\varphi(x_{j} + \frac{1}{2}, t_{k-\frac{1}{2}}))(\delta_{j}^{k+\frac{1}{2}} + \delta_{j}^{k-\frac{1}{2}}).$$

Recalling that the exact solution $\varphi(x, t)$ of the system (1.1) is bounded, i.e. there exists a constant $C_{2}$ such that

$$C_{2} = \max(|\varphi(x, t)|).$$

Then, we derive

$$\|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}\|_{2}^{2} + \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}\|_{2}^{2} - \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}\|_{2}^{2})$$

$$\leq \frac{1}{4}h \sum_{j}(\sigma_{j}^{k+\frac{1}{2}} + O(\tau^{2} + h^{2}))(\delta_{j}^{k+\frac{1}{2}} + \delta_{j}^{k-\frac{1}{2}}) + (C_{1} + C_{2} + \frac{1}{4})(\|\delta_{j}\|_{2}^{2} + \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} + O(\tau^{2} + h^{2})^{2}).$$

Combining (3.8) with (3.11), we get the inequality

$$\|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}\|_{2}^{2} + \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}\|_{2}^{2} - \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} - \|\delta_{j}\|_{2}^{2})$$

$$\leq \tau[\|\varphi^{k+\frac{1}{2}}\|_{2}^{2} + \|\varphi^{k-\frac{1}{2}}\|_{2}^{2} + \|\varphi\|_{2}^{2}] + C_{1}(\|\varphi^{k+\frac{1}{2}}\|_{2}^{2} + \|\varphi^{k-\frac{1}{2}}\|_{2}^{2})$$

$$+ \frac{1}{2}(5C_{1} + C_{2} + 2)(\|\varphi^{k+\frac{1}{2}}\|_{2}^{2} + \|\varphi^{k-\frac{1}{2}}\|_{2}^{2}) + \|\varphi^{k}\|_{2}^{2} + O(\tau^{2} + h^{2})^{2}).$$

Define

$$W^{k} = \|\delta_{j}\|_{2}^{2} + \|\delta_{j}^{k+\frac{1}{2}}\|_{2}^{2} + \|\delta_{j}^{k-\frac{1}{2}}\|_{2}^{2} + \|\varphi^{k+\frac{1}{2}}\|_{2}^{2} + \|\varphi^{k-\frac{1}{2}}\|_{2}^{2} + \|\varphi\|_{2}^{2},$$

and

$$C = \max\{C_{1} + \frac{1}{8}(5C_{1} + C_{2} + 2), (C_{1} + C_{2} + \frac{1}{4})\}.$$
\[ W^N \leq \{(1 + C\tau)W^0 + \tau \sum_{k=1}^{N} \left[ \|r^{k+\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|r^{k-\frac{1}{2}}\|_{\frac{3}{2}}^2 + \frac{1}{4}\|\sigma^k\|_{\frac{3}{2}}^2 + \mathcal{O}(\tau^2 + h^2)^2 \right] \} \exp^{4NC\tau}, \]

where, \(\tau\) is sufficiently small, such that \(\tau \leq \frac{1}{4N}\).

Recalling that the definition of \(W^k\), we have the following estimations:

\[
\begin{align*}
\{\|\delta_x\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}}^2, & \|\delta_x\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}}^2, \|\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}}^2, \|e^{N}\|_{\frac{3}{2}}^2 \} \\
& \leq \{(1 + C\tau)(\|\delta_x\eta^{\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|\delta_x\eta^{\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|\eta^{\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|e^{\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|e^{0}\|_{\frac{3}{2}}^2) \\
& + \tau \sum_{k=1}^{N} \left[ \|r^{k+\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|r^{k-\frac{1}{2}}\|_{\frac{3}{2}}^2 + \frac{1}{4}\|\sigma^k\|_{\frac{3}{2}}^2 + \mathcal{O}(\tau^2 + h^2)^2 \right] \} \exp^{4NC\tau}. \end{align*}
\]

This completes the proof. \(\Box\)

We’d like to emphasize here that the estimates of the solutions are bounded by the initial values and the local truncation errors of solutions.

**Corollary 3.2.** The numerical solutions of the multi-symplectic scheme (2.10)-(2.11) converge to the solutions of system (1.1) with order \(\mathcal{O}(\tau^2 + h^2)\), i.e.

\[ \|e^N\|_{\frac{3}{2}} \leq \mathcal{O}(\tau^2 + h^2), \quad \|\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}} \leq \mathcal{O}(\tau^2 + h^2). \]

**Proof.** Recalling that the initial values \(e^0, e^1, \eta^0, \eta^1\) is at least two order precisely, (3.3) and (3.4), it yields

\[ (3.13) \quad \|\delta_x\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|\delta_x\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}}^2 + \|e^{N+1}\|_{\frac{3}{2}}^2 + \|e^N\|_{\frac{3}{2}}^2 \leq C^*\mathcal{O}(\tau^2 + h^2)^2. \]

Here, \(C^*\) is a bounded constant.

Then we derive

\[ (3.14) \quad \|\delta_x\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}} \leq \mathcal{O}(\tau^2 + h^2), \quad \|\delta_x\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}} \leq \mathcal{O}(\tau^2 + h^2), \quad \|\eta^{N+\frac{1}{2}}\|_{\frac{3}{2}} \leq \mathcal{O}(\tau^2 + h^2), \quad \|e^N\|_{\frac{3}{2}} \leq \mathcal{O}(\tau^2 + h^2). \]

This completes the proof. \(\Box\)

**Lemma 3.3.** (Discrete Sobolev inequality) For any discrete function \(u_h = \{u_j | j = 0, 1, \ldots \}\) in the real axis and for any given \(\varepsilon > 0\), there exists a constant \(K\) dependent on \(\varepsilon\) such that

\[ \|u_h\|_{\infty} \leq \varepsilon \|\delta u_h\| + K\|u_h\|. \]

Utilizing Theorem 3.1, corollary 3.2 and Lemma 3.3, we can obtain the following corollary.

**Corollary 3.4.** The numerical solution of the multi-symplectic scheme (2.10)-(2.11) converges to the solution of the system (1.1) with order \(\mathcal{O}(\tau^2 + h^2)\) in the \(L_{\infty}\) norm for \(u_j^{k+\frac{1}{2}}\).

The results of the numerical experiments are listed in the following section.
4. Numerical experiments

The purpose of this section is to testify the numerical performance of the multi-
symplectic scheme presented in Section 2.

We first concentrate on the solitary-wave solutions of system (1.1)

\[
\varphi(x, t, q) = \frac{3\sqrt{2}}{4\sqrt{1-q^2}} \sech^2 \left( \frac{1}{2\sqrt{1-q^2}} (x - qt) \right) \exp(i(qx + \frac{1-q^2 + q^4}{2(1-q^2)} t)),
\]

\[
u(x, t, q) = \frac{3}{4(1-q^2)} \sech^2 \left( \frac{1}{2\sqrt{1-q^2}} (x - qt) \right),
\]

where, \(0 \leq |q| \leq 1\) indicates the propagating velocity of wave. The initial values

\[
\varphi_0(x) = \varphi(x, 0, q), \quad u_0(x) = u(x, 0, q), \quad v_0(x) = u_t(x, t, q) \big|_{t=0}
\]

are obtained from (4.1) as \(t = 0\).

In the following for the sake of calculation, we set the spatial domain to be considered
is the interval \([-L, L]\), with \(L = 40\), and assume zero boundary conditions, i.e.,

\[
\varphi(-L, t) = \varphi(L, t) = 0, \quad u(-L, t) = u(L, t) = 0.
\]

Furthermore, we set a series of notations as

\[
(error_{\varphi})_k^2 = \left( \frac{h}{2} \sum_j |e_{j+\frac{1}{2}}^k|^2 \right)^{\frac{1}{2}}, \quad (error_u)_k^{k+\frac{1}{2}} = \left( h \sum_j |\eta_{j+\frac{1}{2}}^{k+\frac{1}{2}}|^2 \right)^{\frac{1}{2}},
\]

\[
(error_u)_k^{k+\frac{1}{2}} = \max_j |\eta_{j+\frac{1}{2}}^{k+\frac{1}{2}}|,
\]

(4.2)

to show \(L^2\) errors and \(L^\infty\) error of solutions.

And as stated in section 2, we make use of

\[
(\mathcal{N})^k = \|\varphi^k\|_2^2
\]

to represent the discrete charge conservation law, and let

\[
(error_c)_k^k = (\mathcal{N})^k - (\mathcal{N})^0
\]

(4.3)

to denote the global error of discrete charge conservation law (1.9). And we also denote
the error propagation of discrete total energy as

\[
(error_c)_c^k = \mathcal{E}^k - \mathcal{E}^0.
\]

In order to illustrate the perfect behavior of the multi-symplectic scheme (in short,
SMID), we introduce another implicit non-multi-symplectic methods of order 2 (in short
TRAP), i.e trapezoidal method in both directions of time and space for Klein-Gordon-
Schrödinger equation. Then we can also define all the above notations of errors at the
point \((x_j, t_k)\) for TRAP.
Fig 1 presents the evolution of the numerical solutions $\varphi$, $u$ for solitary-wave, when applying SMID to simulate the solutions $\varphi(x, t, 0.1), u(x, t, 0.1)$ in the time interval $[0, 80]$. If we don’t emphasize in the following, the computing conditions are all as this.

Table 1-2 show the errors and the experimental order of accuracy for SMID and TRAP respectively, in $L^2$ and $L^\infty$ norms at time $T = 10$. Looking at the table 1, it can be observed that the numerical results are consistent to the theoretical results of Theorem 3.1 and its corollaries. The numerical solution $u$ is of order $O(\tau^2 + h^2)$ in both $L^\infty$ norm and $L^2$ norm. And the numerical solution $\varphi$ is also of order $O(\tau^2 + h^2)$ in the $L^2$ norm. Table 2 indicates that the errors of numerical solutions for TRAP are in the same order with SMID.

In Fig 2-4 we show the long time behavior of the numerical solutions $\varphi(x−30, t, 0.1), u(x−30, t, 0.1)$ in the time interval $[0, 300]$ with the temporal stepsize $\tau = 0.05$ and spatial meshgrid-size $h = 0.2$. Fig 2 presents the errors of charge conservation law. As we have proved in the Theorem 2.2 theoretically, SMID preserves the charge conservation law exactly in the scale of $10^{-12}$, and TRAP simulates the the charge conservation law in the scale of $10^{-6}$. Then, we set $\max_j |E_j^{k+\frac{1}{2}}|$ to denote the maximum error of the residual of local energy conservation law (2.3), and its variation is exhibited in Fig 3. The two schemes present the similar numerical behavior and the variation does not have evident increment for a long time. The global errors of total energy presented in Fig 4 are similarly to 3. However, SMID is superior to TRAP in one or two magnitude for both of local energy conservation law and the total energy.

Fig 5 presents errors of discrete charge conservation law with the change of temporal stepsize $\tau$ and spatial meshgrid-size $h$. And when they change, there is no evident difference in the errors for SMID. We can read that SMID preserves the discrete charge conservation law accurately in long time behavior, which can be seen in the diagram are in the scale of $10^{-13}$. Though the graphs exhibit the trend of accumulation of error, the amplifying velocity is extremely slower than the magnitude of the error. But for TRAP it can not conserve the charge conservation law, and the error is reduced to about $\frac{1}{16}$ roughly, as both the meshgrid-size and the time stepsize are reduced to $\frac{1}{2}$.

In Fig 6, we list the maximum errors of local energy conservation law with different temporal stepsize and spatial meshgrid-size. We’d like to point out here from the data in the numerical experiments that the maximum errors approximately reach the magnitude of $O(h^2 + \tau^2)$ for both SMID and TRAP. However, the results of SMID are better than TRAP.

In Fig 7, from the change process it can be observed that the global error of total energy for SMID roughly reaches the magnitude of $O(h^2 + \tau^2)$ at least. If we reduce spatial meshgrid-size or temporal stepsize to half, the error is reduced to about $\frac{1}{16}$ also, similarly, if we reduce both of spatial meshgrid-size and temporal stepsize to half, the error is reduced to about $\frac{1}{16}$ correspondingly. And for TRAP, the global error of total
energy roughly reaches the magnitude of $O(h^4)$, since that the change of error seems not depend on the change of temporal stepsiz. In all, the errors of SMID are about three magnitude better than those of TRAP.

All the graphs in Fig 6-7 exhibit pretty performance, and with oscillation of little amplitude around some equilibrium state, and have no any evident amplification for a long time, which indicate that the multi-symplectic scheme preserves the conservation laws perfectly in long-time behavior, though it can not preserve them exactly.

Now, we turn to the collision of two solitons. The corresponding initial values are given as

\[
\begin{align*}
\varphi_0(x) &= \varphi_0(x - x_1, 0, q_1) + \varphi_0(x - x_2, 0, q_2), \\
u_0(x) &= u_0(x - x_1, 0, q_1) + u_0(x - x_2, 0, q_2), \\
v_0(x) &= \{u_t(x - x_1, t, q_1) + u_t(x - x_2, t, q_2)\}|_{t=0},
\end{align*}
\]

where, $x_1, x_2$ are initial phases, $q_1, q_2$ are propagating velocities of two solitons, respectively.

**Case1:** Collision of two solitons with same speed and opposite direction: $x_1 = -15, x_2 = 15$ and $q_1 = 0.7, q_2 = -0.7$, respectively, and the time interval $[0, 40]$. In this case, we take $\tau = 0.02$ and $h = 0.2$.

Fig 8 shows collision of two solitons in case 1 by applying SMID to the equation at various times. Left column represents the evolution of $\varphi$, and right column represents the evolution of $u$. The two solutions stay at the same position at time around $t = 21.2$ (third row), and the solutions keep in this state after the time and result in fusion, and are accompanied by a series of emission of waves (below row).

In Fig 9, we give the corresponding global errors of charge conservation law, total energy and local energy conservation law for collision of two solitons in case 1. It can be observed that the error of conservation law is in the scale of $10^{-11}$ for SMID, while for TRAP in the scale of $10^{-3}$. And the errors of total energy and local energy conservation law change larger during collision, and exhibit pretty performance with oscillation of little amplitude around some equilibrium state, and have no any evident amplification for a long time after collision for both SMID and TRAP. In general, SMID is better than TRAP.

In Fig 10, the evolutions of numerical solutions in case 1, when applying the multisymplectic scheme, is exhibited.

**Case2:** Collision of two solitons with different speed and opposite direction, for we chose $x_1 = -5, x_2 = 5$ and $q_1 = 0.8, q_2 = -0.6$, respectively, and the time interval $[0, 40]$. In this case, we set $\tau = 0.02$ and $h = 0.2$.

Fig 11 shows collision of two solitons in case 2 at various times. Left column represents the evolution of $\varphi$, and right column represents the evolution of $u$. From the
graphs, we can see that two solitons propagate in the opposite directions, and the two solutions stay at the same position at time $t = 10.46$ (third row), after collision they propagate in their original directions again. The amplitude is becoming bigger for the soliton with larger amplitude originally, and it becomes smaller for the other. It means that the soliton with larger amplitude absorbs part of the other during the collision.

In Fig 12, we list the corresponding global errors of charge conservation law, total energy and local energy conservation law for collision of two solitons in case 2. It can be seen that the error of conservation law is in the scale of $10^{-12}$ for SMID. And others exhibit analogous performance as in Fig 9.

In Fig 13, the evolutions of numerical solutions in case 2 is revealed when applying the multi-symplectic scheme.

| Table 1. Numerical results for SMID |
| --- | --- | --- | --- | --- | --- | --- |
| $\tau \backslash h$ | $L^2\text{error of } \varphi$ | $L^2\text{order of } \varphi$ | $L^2\text{error of } u$ | $L^2\text{order of } u$ | $L^\infty\text{error of } u$ | $L^\infty\text{order of } u$ |
| 0.1\(0.4)
0.05\(0.2)
0.025\(0.1)
0.0125\(0.05)
 | 2.923777e-1
7.595328e-2
1.918203e-2
4.803959e-3 | 1.944649
1.985357
1.997460
4.151569e-4 | 2.722809e-2
6.838523e-3
1.693228e-3
3.373122e-4 | 1.993339
2.013909
2.028047
4.151569e-4 | 2.079337e-2
5.423639e-3
2.028047
3.373122e-4 | 1.938791
2.046831
1.960274
3.373122e-4 |

| Table 2. Numerical results for TRAP |
| --- | --- | --- | --- | --- | --- | --- |
| $\tau \backslash h$ | $L^2\text{error of } \varphi$ | $L^2\text{order of } \varphi$ | $L^2\text{error of } u$ | $L^2\text{order of } u$ | $L^\infty\text{error of } u$ | $L^\infty\text{order of } u$ |
| 0.1\(0.4)
0.05\(0.2)
0.025\(0.1)
0.0125\(0.05)
 | 1.877543e-1
4.832398e-2
1.217280e-2
3.043510e-3 | 1.958035
1.989078
1.999853
4.470766e-4 | 2.868970e-2
7.303376e-3
1.817994e-3
4.70766e-4 | 1.973898
2.006216
2.023754
4.70766e-4 | 1.928425e-2
4.954974e-3
1.212831e-3
3.043510e-3 | 1.960474
2.030499
1.975910
3.043510e-3 |

5. Conclusion

We detailed the properties of the central box scheme, when applied it to the Klein-Gordon-Schrödinger equation. Including the multi-symplectic geometry structure, the preservation of charge, total energy, local energy conservation law and formal energy of the Klein-Gordon-Schrödinger equation itself. The results are as follows.

1. Theorem 2.2 indicates that SMID possesses the discrete charge conservation law under zero boundary conditions.

2. Theorem 2.4 shows that SMID has specific formal energy which can be seen as a discrete version of energy at time $t_{k+\frac{1}{2}}$.

3. Theorem 3.1 gives several estimate of the solutions which is bounded by the initial values of solutions and the local truncation errors. Furthermore, Corollary 3.2 shows that the numerical solutions of SMID approximate the real solutions with order $O(\tau^2 + h^2)$ theoretically.

4. Numerical experiments implies that SMID preserve the charge conservation law exactly, which is consistent to the theoretical result. And from the graphs numerically, it follows that SMID preserves the total energy and the local energy conservation law in the magnitude of $O(h^2\tau^2)$ and $O(\tau^2 + h^2)$, respectively.
Figure 1. The evolution of numerical solutions: Real part (Top), imaginary part (Mid) of $\varphi$, and $u$ (Below), respectively, with $\tau = 0.02$ and $h = 0.2$.

5. From the numerical comparisons, though SMID has the same order as TRAP when approximating the solutions of the Klein-Gordon-Schrödinger equation, it is superior to the other not only on preserving the charge conservation law exactly, but also on preserving the total energy and local energy conservation law in the higher magnitude.
Figure 2. Global errors of charge conservation law, i.e., \((\text{Error}_c)^k\) with temporal stepsize \(\tau = 0.05\) and spatial mesh-grid \(h = 0.2\), respectively. SMID(Left column); TRAP(Right column).

Figure 3. The maximum errors of local energy conservation law, i.e., \(\max_j |E_{k+\frac{1}{2}}^j|\) with temporal stepsize \(\tau = 0.05\) and spatial mesh-grid \(h = 0.2\), respectively. SMID(Left column); TRAP(Right column).

Figure 4. Global errors of total energy, i.e., \((\text{Error}_e)^k\) with temporal stepsize \(\tau = 0.05\) and spatial mesh-grid \(h = 0.2\), respectively. SMID(Left column); TRAP(Right column).

All these are the most interesting things that motivate us to discuss the multisymplectic methods here.

**Acknowledgement:** Authors would like to thank Dr. Kong for useful discussions.
Figure 5. Global errors of charge conservation law, i.e., \((Error_c)^k\) with different spatial and temporal stepsize: above for \(\tau = 0.01\) and \(h = 0.1\); Mid for \(\tau = 0.02\) and \(h = 0.2\); Below for \(\tau = 0.04\) and \(h = 0.4\). SMID(Left column); TRAP(Right column).

REFERENCES

Figure 6. Maximum errors of local energy conservation law, i.e., \( \max_{j} |E^{k+\frac{1}{2}}_{j}| \) with different spatial and temporal stepsizes: Above for \( \tau = 0.01 \) and \( h = 0.1 \); Second for \( \tau = 0.01 \) and \( h = 0.2 \); Mid for \( \tau = 0.02 \) and \( h = 0.2 \); Fourth for \( \tau = 0.04 \) and \( h = 0.2 \); Below for \( \tau = 0.04 \) and \( h = 0.4 \). SMID(Left column); TRAP(Right column).
Figure 7. Global error of total energy, i.e., $(\text{Error}_e)^k$ with different spatial and temporal stepsizes: Above for $\tau = 0.01$ and $h = 0.1$; Second for $\tau = 0.01$ and $h = 0.2$; Mid for $\tau = 0.02$ and $h = 0.2$; Fourth for $\tau = 0.04$ and $h = 0.2$; Below for $\tau = 0.04$ and $h = 0.4$. SMID(Left column); TRAP(Right column).
Figure 8. The evolution of numerical solutions for soliton-soliton collision with $q_1 = 0.7, q_2 = -0.7, x_1 = -15, x_2 = 15$ at various times. Left column: $|\varphi|$; Right column: $u$. 
Figure 9. Global error of charge conservation, total energy and local energy conservation law for soliton-soliton collision with $q_1 = 0.7, q_2 = -0.7, x_1 = -15, x_2 = 15$, with spatial stepsize $\tau = 0.02$ and temporal stepsize $h = 0.2$. SMID (Left column); TRAP (Right column).

Figure 10. Graphs of three dimensions of numerical solutions for soliton-soliton collision with $q_1 = 0.7, q_2 = -0.7, x_1 = -15, x_2 = 15$. Left: $|\varphi|$; Right: $u$. 

STATE KEY LABORATORY OF SCIENTIFIC AND ENGINEERING COMPUTING, INSTITUTE OF COMPUTATIONAL MATHEMATICS AND SCIENTIFIC/ENGINEERING COMPUTING, ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, CHINESE ACADEMY OF SCIENCES, P.O. BOX 2719, BEIJING 100080, P. R. CHINA
E-mail address: hjl@lsec.cc.ac.cn

1STATE KEY LABORATORY OF SCIENTIFIC AND ENGINEERING COMPUTING, INSTITUTE OF COMPUTATIONAL MATHEMATICS AND SCIENTIFIC/ENGINEERING COMPUTING, ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, CHINESE ACADEMY OF SCIENCES, P.O. BOX 2719, BEIJING 100080, P. R. CHINA

2GRADUATE SCHOOL OF THE CHINESE ACADEMY OF SCIENCES, BEIJING 100080, P. R. CHINA
E-mail address: jiangss@lsec.cc.ac.cn

1STATE KEY LABORATORY OF SCIENTIFIC AND ENGINEERING COMPUTING, INSTITUTE OF COMPUTATIONAL MATHEMATICS AND SCIENTIFIC/ENGINEERING COMPUTING, ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, CHINESE ACADEMY OF SCIENCES, P.O. BOX 2719, BEIJING 100080, P. R. CHINA

2GRADUATE SCHOOL OF THE CHINESE ACADEMY OF SCIENCES, BEIJING 100080, P. R. CHINA
E-mail address: lichun@lsec.cc.ac.cn
Figure 11. The evolution of numerical solutions for soliton-soliton collision with $q_1 = 0.8, q_2 = -0.6, x_1 = -10, x_2 = 5$ at various times. Left column: $|\psi|$; Right column: $u$. 
Figure 12. Global error of charge conservation, total energy and local energy conservation law for soliton-soliton collision with, $q_1 = 0.8, q_2 = -0.6, x_1 = -10, x_2 = 5$, with spatial stepsize $\tau = 0.02$ and temporal stepsize $h = 0.2$. SMID (Left column); TRAP (Right column).
Figure 13. Graphs of three dimensions of numerical solutions for soliton-soliton collision with $q_1 = 0.8, q_2 = -0.6, x_1 = -10, x_2 = 5$. Left: $|\varphi|$; Right: $u$. 