Numer. Math. (2005)
DOI 10.1007/s00211-005-0668-4

Numerische
Mathematik

Li-Qun Cao

Multiscale asymptotic expansion and finite element methods for the mixed boundary value problems of second order elliptic equation in perforated domains

Received: 24 December 2003 / Revised: 18 March 2005

Abstract In this paper, we will discuss the mixed boundary value problems for the second order elliptic equation with rapidly oscillating coefficients in perforated domains, and will present the higher-order multiscale asymptotic expansion of the solution for the problem, which will play an important role in the numerical computation . The convergence theorems and their rigorous proofs will be given. Finally a multiscale finite element method and some numerical results will be presented.

Keywords Homogenization · Multiscale asymptotic expansion· Elliptic equation of second order with rapidly oscillating coefficients· Perforated domain· The mixed boundary value problem· Boundary layer

AMS Subject Classifications: 65F10, 35B50

1 Introduction

Let ω be an unbounded domain of R^n with a 1-periodic structure, i.e. ω is invariant under the shifts by any $z = (z_1, \dots z_n) \in Z^n$, following Oleinik's notation (see, [18], pp. 42).

Suppose that ω satisfies the following conditions:

(B_1) ω is a smooth unbounded domain of R^n with a 1-periodic structure.

(B_2) The cell of periodicity $\omega \cap Q$, $Q = (0, 1)^n$ is a domain with a Lipschitz boundary.

This work is Supported by National Natural Science Foundation of China (grant # 10372108, # 90405016), and Special Funds for Major State Basic Research Projects(grant # TG2000067102)

L.-Q. Cao
Institute of Computational Mathematics and Science-Engineering Computing,
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences,
P.O.Box 2719, Beijing, 100080, P.R.China;
E-mail: clq@lsec.cc.ac.cn

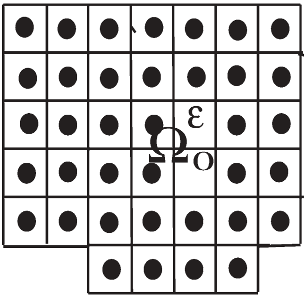


Fig. 1.1 Subdomain Ω_0^ε

(B_3) The set $Q \setminus \bar{\omega}$ and the intersection of $Q \setminus \bar{\omega}$ with the δ_0- neighborhood ($\delta_0 < \frac{1}{4}$) of ∂Q consist of a finite number of Lipschitz domains separated from each other and from the edges of the cube Q by a positive distance.

A domain Ω^ε has the form: $\overline{\Omega}^\varepsilon = \overline{\Omega}_0^\varepsilon \cup (\overline{\Omega} \setminus \Omega_0)$, where Ω is a bounded Lipschitz convex domain of R^n without cavities, $\overline{\Omega}_0 = \cup_{z \in T_\varepsilon} \varepsilon(z + \bar{Q}) \subset \Omega$, $\overline{\Omega}_0^\varepsilon = \overline{\Omega}_0 \cap \varepsilon\bar{\omega}$ is shown in Fig. 1.1, T_ε is the subset of Z^n consisting of all z such that $\varepsilon(z + Q) \subset \Omega$, $\rho(\varepsilon(z + Q), \partial\Omega) \geq \varepsilon$, ε is a small parameter. The domain Ω_1 of the boundary layer has the form $\Omega_1 = \Omega \setminus \overline{\Omega}_0$ as shown in Fig. 1.2. The boundary $\partial\Omega^\varepsilon$ of a domain Ω is the union of $\partial\Omega$ and the surface $S_\varepsilon \subset \Omega$ of the cavities, $S_\varepsilon = \partial\Omega^\varepsilon \cap \Omega$.

Consider the boundary value problem of second order elliptic equation with rapidly oscillating coefficients as follows

$$\begin{cases} \mathcal{L}_\varepsilon u^\varepsilon \equiv -\dfrac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x)}{\partial x_j}) = f(x) \quad \text{in} \quad \Omega^\varepsilon \\ \sigma_\varepsilon(u^\varepsilon) \equiv -\nu_i a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon}{\partial x_j} = 0, \quad \text{on} \quad S_\varepsilon \\ u^\varepsilon(x) = \bar{u}(x), \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where $f \in H^{-1}(\Omega)$, $\bar{u} \in H^{1/2}(\partial\Omega)$ are some given functions.

Suppose that

(A_1) $a_{ij}(\frac{x}{\varepsilon}) = a_{ij}(\xi)$, $\xi = \frac{x}{\varepsilon}$, are 1-periodic functions in ξ.

(A_2) $\gamma_0|\eta|^2 \leq a_{ij}(\xi)\eta_i\eta_j \leq \gamma_1|\eta|^2, \quad \gamma_0, \gamma_1 > 0, \quad \xi \in Q \cap \omega \quad \forall(\eta_1, \dots \eta_n) \in R^n.$

(A_3) $a_{ij}(\xi) = a_{ji}(\xi)$, $a_{ij} \in L^\infty(R^n)$.

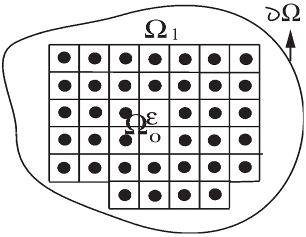


Fig. 1.2 Boundary layer Ω_1

Some homogenization results and their proofs for the mixed boundary value problems of second order elliptic equation in a perforated domain were proved by Cioranescu et al.(cf. [7]) in the situation: with the homogeneous Neumann's conditions (or Fourier's ones) on the surface of holes, and with the Dirichlet's boundary conditions on the outer boundary part, under the assumption that the holes in the reference cell Q do not intersect the boundary ∂Q . Allaire and Murat (cf. [2]) derived the convergence theorem for the above problems concerning the case of nonisolated holes. Oleinik (cf. [18]) studied homogenization for the mixed problem for the elastic systems of second order equations in a perforated domain, and gave the convergence results with order $O(\varepsilon^{1/2})$ for the first order corrector. Conca and Natesan [8] studied elliptic PDEs with oscillating coefficients, and determined the numerical solution through asymptotic expansion, and article [9] which presented a numerical method for homogenization problem through an integral representation formula, via Bloch decomposition.

It should be pointed out that the homogenization method only describes the asymptotic behavior of the solution for problem (1.1), as $\varepsilon \to 0$. However, in many real problems, ε does not approach to zero, and in fact ε is not very small. For example, when we consider 3-D woven structures of composite materials, even if we select ε in the range $1/10 \sim 1/100$, it is extremely difficult to calculate directly some mechanical fields such as stresses and strains. The number of elements is about $10^6 \sim 10^{10}$. Therefore we would like to say that it is necessary to seek higher-order asymptotic methods.

Lions (cf. [17]) gave the higher-order asymptotic expansion of the solution for the Dirichlet boundary value problem of the equation $-\triangle u^\varepsilon = f(x)$ in a perforated domain Ω^ε provided that $f \in C_0^\infty(\Omega)$. Oleinik et al. (cf. [18]) got rid of the assumption of $f \in C_0^\infty(\Omega)$ and presented the higher-order asymptotic expansion of the solution for the Dirichlet boundary value problem of the elastic systems of second order equations in a perforated domain Ω^ε by constructing properly the boundary layer. Observing carefully their proofs, it is not difficult to see the crucial point that Friedrichs-Poincaré inequality is true in the unit cell, due to the Dirichlet boundary conditions on the surfaces of holes. However, one can derive on error estimate which is only of order $\varepsilon^{1/2}$ in a domain without any holes or in a perforated domain satisfying Neumann boundary conditions on the surfaces of holes (see [7,18]). It should be emphasized that the above method(i.e. periodic boundary conditions on the boundary of the unit cube $Q = (0, 1)^n$) maybe does not work for a subdivided periodic structure, due to the first-order or higher-order asymptotic solutions $u_s^\varepsilon \notin H^1(\Omega^\varepsilon)$ in latter cases. The crucial idea of the method proposed in this paper is to use homogeneous Dirichlet boundary conditions on the boundary of the unit cube $Q = (0, 1)^n$). Specially it is suitable for a subdivided periodic domain. In the next sections, we would like to compare the advantages and shortcomings of periodic boundary conditions compared to the homogeneous Dirichlet boundary conditions on ∂Q.

Our goals in this paper are to give the higher-order multiscale asymptotic expansion of the solution for the mixed boundary value problems of second order elliptic equation in a perforated domain, and to present a multiscale finite element method. The rigorous proofs of all theoretical results will be given. Finally, some numerical results will be reported.

Throughout the paper the Einstein summation convention on repeated indices is adopted, and C denotes a positive constant without distinction.

2 A Formal Asymptotic Expansion and the Construction of Boundary Layer

To begin with, let us give a formal multiscale asymptotic expansion of the solution for considering problem (1.1). Set formally

$$u^{\varepsilon}(x) \cong \sum_{l=0}^{+\infty} \varepsilon^{l} \sum_{\alpha_1,\dots,\alpha_l=1}^{n} N_{\alpha_1\dots\alpha_l}(\xi) D^{\alpha} u^{0}(x) \tag{2.1}$$

We next give the definitions of $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\dots N_{\alpha_1\cdots\alpha_l}(\xi)$, $\alpha_j = 1, 2, \dots, n$, $1 \le j \le l, l \ge 1$:

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)\Big), \text{in} \quad Q \cap \omega \\ \sigma_{\xi}(N_{\alpha_1}) = -\nu_i a_{i\alpha_1}(\xi), \quad \text{on} \quad Q \cap \partial\omega \\ N_{\alpha_1}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{2.2}$$

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi)\Big) \\ \qquad -a_{\alpha_1 j}(\xi)\dfrac{\partial N_{\alpha_2}(\xi)}{\partial \xi_j} - a_{\alpha_1\alpha_2}(\xi) + \hat{a}_{\alpha_1\alpha_2}, \quad \text{in} \quad Q \cap \omega \\ \sigma_{\xi}(N_{\alpha_1\alpha_2}) = -\nu_i a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi), \quad \text{on} \quad Q \cap \partial\omega \\ N_{\alpha_1\alpha_2}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{2.3}$$

where

$$\hat{a}_{ij} = \frac{1}{|Q \cap \omega|} \int\limits_{Q\cap\omega} (a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial N_j(\xi)}{\partial \xi_k})d\xi \tag{2.4}$$

For $\langle \alpha \rangle = l \ge 3$

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1\dots\alpha_l}(\xi)}{\partial \xi_j}) = -\dfrac{\partial}{\partial \xi_i}(a_{i\alpha_1}(\xi)N_{\alpha_2\dots\alpha_l}(\xi)) \\ \qquad -a_{\alpha_1 j}(\xi)\dfrac{\partial N_{\alpha_2\dots\alpha_l}(\xi)}{\partial \xi_j} - a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\dots\alpha_l}(\xi), \quad \text{in} \quad Q \cap \omega \\ \sigma_{\xi}(N_{\alpha_1\dots\alpha_l}) = -\nu_i a_{i\alpha_1}(\xi)N_{\alpha_2\dots\alpha_l}(\xi), \quad \text{on} \quad Q \cap \partial\omega \\ N_{\alpha_1\dots\alpha_l}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{2.5}$$

Remark 2.1 Existence and uniqueness of $N_{\alpha_1}(\xi), \dots N_{\alpha_1\dots\alpha_l}(\xi)$ can be easily established by induction with respect to l due to the uniform elliptic condition (A_2), Poincaré -Friedrichs' inequality and Lax-Milgram's lemma.They are then extended to the unbounded domain ω in 1-periodicity. Without confusion we continue to use $N_{\alpha_1}(\xi), \dots N_{\alpha_1\dots\alpha_l}(\xi)$ to denote the periodic extensions.

Remark 2.2 It should be mentioned that the periodic functions $N_{\alpha_1}(\xi),\ldots N_{\alpha_1\ldots\alpha_l}(\xi)$ defined in this paper are different from those of some classical homogenization methods (see, e.g. [2,4,7,13,15,18]), due to different boundary conditions on the boundary ∂Q of the unit cell $Q\cap\omega$. Under the assumptions of periodic boundary conditions, we know that periodic functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$ and their normal derivatives are continuous on the interface $\bigcup_{z\in T_\varepsilon}\partial E_z$, where $\bar{E}_z=\varepsilon(z+\bar{Q})$. However, the normal derivatives of periodic functions with homogeneous Dirichlet boundary conditions on ∂Q, generally speaking, are not continuous on the interface. In section 4, we shall prove that the normal derivatives of $N_{\alpha_1}(\xi),\ldots N_{\alpha_1\ldots\alpha_3}(\xi)$ are continuous in a sense on the interface ∂Q under some assumptions.

The homogenized equation associated with (1.1) is the following:

$$\begin{cases}\widehat{\mathcal{L}}u^0(x)\equiv-\dfrac{\partial}{\partial x_i}(\hat{a}_{ij}\dfrac{\partial u^0(x)}{\partial x_j})=f(x) \quad \text{in} \quad \Omega\\ u^0(x)=\bar{u}(x), \quad \text{on} \quad \partial\Omega\end{cases}\tag{2.6}$$

Remark 2.3 In the next section, it can be proven that the homogenized coefficients tensor of our method is the same as that of the classical homogenization method under some assumptions. Furthermore, it is easy to see that $\widehat{\mathcal{L}}$ is a symmetric positive-definite operator, and there exists a unique weak solution of problem (2.6).

For $s\geq 1$, define

$$u_s^\varepsilon(x)=u^0(x)+\sum_{l=1}^{s}\varepsilon^l\sum_{\alpha_1,\ldots,\alpha_l=1}^{n}N_{\alpha_1\ldots\alpha_l}(\xi)D^\alpha u^0(x)\tag{2.7}$$

Generally speaking, $u_s^\varepsilon(x)$ does not satisfy the boundary conditions on $\partial\Omega$ for the solution $u^\varepsilon(x)$ in general domains. In order to catch the boundary effects on the solution $u^\varepsilon(x)$, we need to construct properly a boundary layer so as to improve the accuracy of the approximation $u_s^\varepsilon(x)$, which is not an easy problem(cf. [16, pp.121]).

Now define a boundary layer in such a way:

$$\begin{cases}\mathcal{L}_\varepsilon u^{\varepsilon,b}\equiv-\dfrac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^{\varepsilon,b}}{\partial x_j})=f(x) \quad x\in\Omega_1\\ u^{\varepsilon,b}(x)=u^0(x) \quad x\in\Gamma^*=\partial\Omega_0\cap\partial\Omega_1\\ u^{\varepsilon,b}(x)=\bar{u}(x) \quad x\in\partial\Omega\end{cases}\tag{2.8}$$

where Ω_1 is a bounded domain as shown in Fig. 1.2, and $u^0(x)$ is the unique solution of the homogenized problem (2.6).

As a whole the approximate solution $U_s^\varepsilon(x)$ of the exact solution $u^\varepsilon(x)$ of problem (1.1) can be written as follows:

$$U_s^\varepsilon(x)=\begin{cases}u_s^\varepsilon(x)=u^0(x)+\sum\limits_{l=1}^{s}\varepsilon^l\sum\limits_{\alpha_1,\ldots,\alpha_l=1}^{n}N_{\alpha_1\ldots\alpha_l}(\xi)D^\alpha u^0(x) \quad x\in\overline{\Omega}_0^\varepsilon\\ u^{\varepsilon,b}(x) \quad x\in\Omega_1=\Omega\setminus\overline{\Omega}_0\end{cases}\tag{2.9}$$

Remark 2.4 According to the definitions of $N_{\alpha_1\ldots\alpha_l}(\xi)$, $\alpha_j = 1, 2, \ldots, n$, $j = 1, 2, \ldots l$, $l \geq 1$, one can derive $u^{\varepsilon,b}(x)|_{\partial\Omega_0\cap\partial\Omega_1} = u_s^{\varepsilon}(x)|_{\partial\Omega_0\cap\partial\Omega_1}$, and consequently $U_s^{\varepsilon} \in H^1(\Omega^{\varepsilon}, \partial\Omega)$. But, generally speaking, $[\frac{\partial U_s^{\varepsilon}}{\partial n}]|_{\partial\Omega_0\cap\partial\Omega_1} \neq 0, \quad s \geq 1$, where $[\frac{\partial U_s^{\varepsilon}}{\partial n}]|_{\partial\Omega_0\cap\partial\Omega_1}$ denotes the jump of normal derivatives of $U_s^{\varepsilon}(x)$ on the interface $\partial\Omega_0 \cap \partial\Omega_1$.

3 Some Preliminary Results

Just as it has been stated in Remark 2.2, the normal derivatives of cell functions with the homogeneous Dirichlet boundary conditions on ∂Q, generally speaking, are not continuous on ∂Q. In this section, we shall discuss this kind of problem and give some useful results.

To start with, let us make some assumptions on the holes $Q \setminus \bar{\omega}$ and the coefficients $a_{ij}(\xi)$:

$[HS]$: The holes $Q \setminus \bar{\omega}$ are symmetric with respect to the middle hyperplanes $\Delta_1, \ldots, \Delta_n$ of the unit cube Q as shown in Fig. 3.1.

[MS]: $a_{ii}, i = 1, 2, \ldots, n$ are symmetric, and $a_{ij}, i \neq j$ are anti-symmetric with respect to middle superplanes $\Delta_1, \ldots, \Delta_n$. In particular, $a_{ij} = 0$, if $\quad i \neq j$.

If $a_{ij}(\xi) \in C^0(\overline{Q \cap \omega})$, we can find a sequence of functions $a_{ij}^{(\beta)}(\xi) \in C^{\infty}(Q \cap \omega)$ satisfy the following properties:

(i) $a_{ij}^{(\beta)}(\xi)$ are 1-periodic functions with respect to ξ;

(ii) $a_{ij}^{(\beta)}(\xi) = a_{ji}^{(\beta)}(\xi)$;

(iii) $\mu_0|\eta|^2 \leq a_{ij}^{(\beta)}\eta_i\eta_j \leq \mu_1|\eta|^2, \quad a.e. \quad \xi \in \omega, \quad \mu_0, \mu_1 > 0, \quad |\eta|^2 = (\sum_{i=1}^n \eta_i^2)^{1/2}$.
and

$$\|a_{ij}^{(\beta)} - a_{ij}\|_{C^0(\overline{Q\cap\omega})} \to 0, \text{ as } \quad \beta \to \infty$$

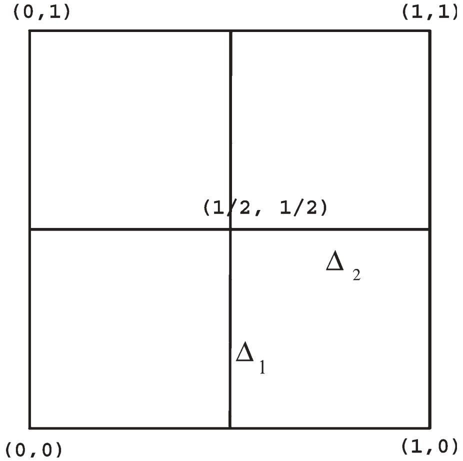


Fig. 3.1 The symmetry of Q

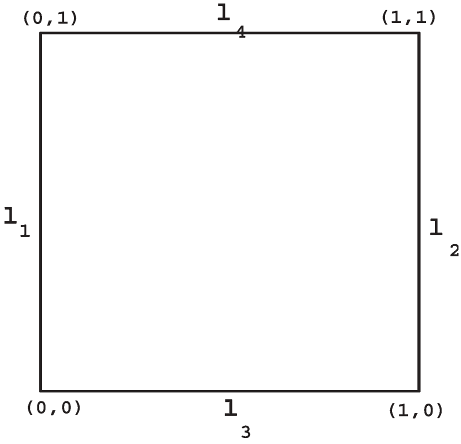


Fig. 3.2 The sides of Q

Then it is standard to verify that(see, [4], pp.103):

$$\|u^{(\beta),\varepsilon} - u^{\varepsilon}\|_{1,\Omega^{\varepsilon}} \leq C \cdot \sup_{i,j} \|a_{ij}^{(\beta)} - a_{ij}\|_{C^0(\overline{Q\cap\omega})}$$

where $u^{(\beta),\varepsilon}$ is the solution of problem (1.1) with the coefficients $a_{ij}^{(\beta)}(\frac{x}{\varepsilon})$.

Assume that the following results proven when the coefficients are C^∞ without loss of generality. For simplicity, we next consider 2-D problems, in practice, the corresponding results are also valid in higher dimensional cases ($n \geq 3$).

Lemma 3.1 *Let $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi), \ldots N_{\alpha_1\ldots\alpha_l}(\xi), \alpha_j = 1, 2, j = 1, 2, \ldots, l$ be the solutions of problems (2.2),(2.3) and (2.5), respectively. Under the assumptions (HS),* **(MS)***, and $a_{ij}(\xi) \in C^0(\overline{Q\cap\omega})$, then one can prove that $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\cdots N_{\alpha_1\cdots\alpha_l}(\xi)$, $\alpha_j = 1, 2$, $j = 1, 2, \cdots, l$ are symmetric or anti-symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2$ of the unit cube Q, e.g.*
$N_1(\xi_1, \xi_2) = -N_1(1-\xi_1, \xi_2), \qquad N_1(\xi_1, \xi_2) = N_1(\xi_1, 1-\xi_2);$
$N_2(\xi_1, \xi_2) = N_2(1-\xi_1, \xi_2), \qquad N_2(\xi_1, \xi_2) = -N_2(\xi_1, 1-\xi_2);$
$N_{11}(\xi_1, \xi_2) = N_{11}(1-\xi_1, \xi_2), \qquad N_{11}(\xi_1, \xi_2) = N_{11}(\xi_1, 1-\xi_2);$
$N_{22}(\xi_1, \xi_2) = N_{22}(1-\xi_1, \xi_2), \qquad N_{22}(\xi_1, \xi_2) = N_{22}(\xi_1, 1-\xi_2);$
$N_{12}(\xi_1, \xi_2) = -N_{12}(1-\xi_1, \xi_2), \qquad N_{12}(\xi_1, \xi_2) = -N_{12}(\xi_1, 1-\xi_2);$
$N_{21}(\xi_1, \xi_2) = -N_{21}(1-\xi_1, \xi_2), \qquad N_{21}(\xi_1, \xi_2) = -N_{21}(\xi_1, 1-\xi_2).$

Proof Lemma 3.1 is a straightforward consequence from the definitions of cell functions and conditions **(HS)** and **(MS)**.

One of our aims in this paper is to prove that the normal derivatives $\sigma_\xi(N_{\alpha_1})$, $\sigma_\xi(N_{\alpha_1\alpha_2})$, $\sigma_\xi(N_{\alpha_1\alpha_2\alpha_3})$ of cell functions are continuous in a sense on the interface ∂Q under the assumptions of **(HS)** and **(MS)**, by means of Fourier analysis. However, one cannot directly use Fourier transformation in the unit cell $Q \cap \omega$ with some holes. To this end, from conditions $(A_1) - (A_3)$, $(B_1) - (B_3)$, (HS) and (MS), one can extend the functions $a_{ij}(\xi)$ in $Q \cap \omega$ as the functions $a_{ij}^*(\xi)$ in the unit cube $Q = (0, 1)^2$ such that they satisfy the conditions $(A_1) - (A_3)$, (MS), and $a_{ij}^*(\xi) \in C^0(\bar{Q})$.

In practice, $a^*_{ij}(\xi)$ can be given in such a way:

$$a^*_{ij}(\xi) = \begin{cases} a_{ij}(\xi) & \xi \in Q \cap \omega \\ \phi^{\delta}_{ij}(\xi) & \xi \in \mathcal{V}_\delta \\ \delta^{1/8} & \xi \in \mathcal{V}_0 \end{cases} \tag{3.1}$$

where $\phi^{\delta}_{ij}(\xi) \in C^\infty(\mathcal{V}_\delta)$, and $\|\phi^{\delta}_{ij}\|_{L^\infty(\mathcal{V}_\delta)} \leq M$, and $\mathcal{V}_\delta = \{\xi \in (Q \setminus \omega) : dist(\xi, \partial\omega) \leq \delta\}$, and $\mathcal{V}_0 = \{\xi \in (Q \setminus \omega) : dist(\xi, \partial\omega) \geq \delta\}$, and δ is a sufficiently small positive number.

Assume that $N^*_{\alpha_1}(\xi), N^*_{\alpha_1\alpha_2}(\xi), \cdots, N^*_{\alpha_1\cdots\alpha_l}(\xi)$ satisfy the following equations:

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a^*_{ij}(\xi)\dfrac{\partial N^*_{\alpha_1}(\xi)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a^*_{i\alpha_1}(\xi)\Big), \quad \text{in} \quad Q \\ N^*_{\alpha_1}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{3.2}$$

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a^*_{ij}(\xi)\dfrac{\partial N^*_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a^*_{i\alpha_1}(\xi)N^*_{\alpha_2}(\xi)\Big) \\ \qquad -a^*_{\alpha_1 j}(\xi)\dfrac{\partial N^*_{\alpha_2}(\xi)}{\partial \xi_j} - a^*_{\alpha_1\alpha_2}(\xi) + \hat{a}^*_{\alpha_1\alpha_2}, \quad \text{in} \quad Q \\ N^*_{\alpha_1\alpha_2}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{3.3}$$

where

$$\hat{a}^*_{\alpha_1\alpha_2} = \frac{1}{|Q|}\int_Q \Big(a^*_{\alpha_1\alpha_2}(\xi) + a^*_{\alpha_1 j}(\xi)\frac{\partial N^*_{\alpha_2}(\xi)}{\partial \xi_j}\Big)d\xi \tag{3.4}$$

For $\langle\alpha\rangle = l \geq 3$

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}(a^*_{ij}(\xi)\dfrac{\partial N^*_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_j}) = -\dfrac{\partial}{\partial \xi_i}(a^*_{i\alpha_1}(\xi)N^*_{\alpha_2\cdots\alpha_l}(\xi)) \\ \quad -a^*_{\alpha_1 j}(\xi)\dfrac{\partial N^*_{\alpha_2\cdots\alpha_l}(\xi)}{\partial \xi_j} - a^*_{\alpha_1\alpha_2}(\xi)N^*_{\alpha_3\cdots\alpha_l}(\xi), \quad \text{in} \quad Q \\ N^*_{\alpha_1\cdots\alpha_l}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases} \tag{3.5}$$

Similarly to Lemma 3.1, one can easily check that $N^*_{\alpha_1}(\xi), \cdots, N^*_{\alpha_1\cdots\alpha_l}(\xi)$ are symmetric or anti-symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2$ of the unit cube Q.

From (3.2), if let

$$\Lambda_{11} = \Big(a^*_{11}(\xi) + \sum_{j=1}^{2} a^*_{1j}(\xi)\frac{\partial N^*_1(\xi)}{\partial \xi_j}\Big)$$

$$\Lambda_{21} = \Big(a^*_{21}(\xi) + \sum_{j=1}^{2} a^*_{2j}(\xi)\frac{\partial N^*_1(\xi)}{\partial \xi_j}\Big),$$

then it can be rewritten as follows:

$$\frac{\partial}{\partial \xi_1}(\Lambda_{11}) + \frac{\partial}{\partial \xi_2}(\Lambda_{21}) = 0 \tag{3.6}$$

Set $v(\xi) = (e^{i2\pi m_1\xi_1} - 1)\cdot(e^{i2\pi m_2\xi_2} - 1),\ m_1 \neq 0, m_2 \neq 0$. It is obvious that $v \in H_0^1(Q)$.

The variational formulation of equation (3.6) is the following:

$$i2\pi\sum_{k=1}^{2} m_k \int\limits_Q \Lambda_{k1}(\xi)(e^{i2\pi m\cdot\xi} - e^{i2\pi m_k\xi_k})d\xi = 0, \quad m\cdot\xi = m_1\xi_1 + m_2\xi_2$$

and consequently

$$m_1(c_{m_1m_2}^{(1)} - c_{m_10}^{(1)}) + m_2(d_{m_1m_2}^{(1)} - d_{0m_2}^{(1)}) = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{3.7}$$

where $c_{m_1m_2}^{(1)} = \int\limits_Q \Lambda_{11}e^{i2\pi m\cdot\xi}d\xi, \qquad c_{m_10}^{(1)} = \int\limits_Q \Lambda_{11}e^{i2\pi m_1\xi_1}d\xi,$

$d_{m_1m_2}^{(1)} = \int\limits_Q \Lambda_{21}e^{i2\pi m\cdot\xi}d\xi, \qquad d_{0m_2}^{(1)} = \int\limits_Q \Lambda_{21}e^{i2\pi m_2\xi_2}d\xi.$

Lemma 3.2 *Under the assumptions of Lemma 3.1, then it can be proven that the corresponding Fourier series of the function* $F(\xi_1,\xi_2) = \Lambda_{11}(\xi_1,\xi_2)$ *is absolutely uniform convergence on* $[0, 1] \times [0, 1]$.

Proof Set $F(\xi_1,\xi_2) = \Lambda_{11}(\xi_1,\xi_2) = (a_{11}^*(\xi) + \sum_{j=1}^2 a_{1j}^*(\xi)\frac{\partial N_1^*(\xi)}{\partial\xi_j})$.

By means of Theorem 6.17, Theorem 6.18 of [10], and Theorem 7.2 of [11], the validity of the following procedure can be guaranteed under the assumption of regularity of the coefficients.

For $m_1 \neq 0$, and $m_2 \neq 0$, $\tilde{c}_{m_1m_2}^{(1)}$ denotes the m_1, m_2 th Fourier coefficient of $F_{\xi_2\xi_1}$; thus

$$\tilde{c}_{m_1m_2}^{(1)} = \int_0^1\int_0^1 F_{\xi_2\xi_1}e^{i2\pi m\cdot\xi}d\xi$$

If we integrate by parts with respect to ξ_2, holding ξ_1 fixed, then by periodicity we obtain

$$\tilde{c}_{m_1m_2}^{(1)} = -i2\pi m_2\int_0^1\int_0^1 F_{\xi_1}e^{i2\pi m\cdot\xi}d\xi$$

Integrating by parts again yields

$$\tilde{c}_{m_1m_2}^{(1)} = -4\pi^2 m_1m_2c_{m_1m_2}^{(1)} \tag{3.8}$$

If we let $\tilde{c}_{0m_2}$, for $m_2 \neq 0$, denote the m_2-th Fourier coefficient of F_{ξ_2}, then $\tilde{c}_{0m_2} = -i2\pi m_2c_{0m_2}$, and similarly, for $m_1 \neq 0$, we have $\tilde{c}_{m_10} = -i2\pi m_1c_{m_10}$. Finally, we define $\tilde{c}_{00}$ to 0. Bessel's inequality applied to F_{ξ_1}, F_{ξ_2}, and $F_{\xi_1\xi_2}$ implies that

$$\sum_{m_1,m_2=-\infty}^{+\infty}|\tilde{c}_{m_1m_2}|^2 \leq \int_0^1\int_0^1[F_{\xi_1}^2 + F_{\xi_2}^2 + F_{\xi_2\xi_1}^2]d\xi_1d\xi_2 \tag{3.9}$$

and by using Cauchy inequality, we obtain:

$$\begin{aligned}\sum|c^{(1)}_{m_1m_2}| &= |c^{(1)}_{00}|+\sum{}'|\tilde{c}^{(1)}_{m_1m_2}||\frac{1}{m_1m_2}|+\sum{}'|\tilde{c}^{(1)}_{m_10}||\frac{1}{m_1}|+\sum{}'|\tilde{c}^{(1)}_{0m_2}||\frac{1}{m_2}|\\ &\le |c^{(1)}_{00}|+\Big(\sum{}'|\tilde{c}^{(1)}_{m_1m_2}|^2\Big)^{1/2}\Big(\sum{}'\frac{1}{m_1^2m_2^2}\Big)^{1/2}\\ &\quad+\Big(\sum{}'|\tilde{c}^{(1)}_{m_10}|^2\Big)^{1/2}\Big(\sum{}'\frac{1}{m_1^2}\Big)^{1/2}\\ &\quad+\Big(\sum{}'|\tilde{c}^{(1)}_{0m_2}|^2\Big)^{1/2}\Big(\sum{}'\frac{1}{m_2^2}\Big)^{1/2} \qquad (3.10)\end{aligned}$$

Because of (3.9), the last sum above is finite. Consequently, we complete the proof of Lemma 3.2.

Proposition 3.1 *Under the assumptions of Lemma 3.1, we have*

$$m_1c^{(1)}_{m_1m_2}+m_2d^{(1)}_{m_1m_2}=0, \quad m_1\neq 0, m_2\neq 0 \qquad (3.11)$$

Furthermore

$$m_1c^{(1)}_{m_10}+m_2d^{(1)}_{0m_2}=0, \quad m_1\neq 0, m_2\neq 0 \qquad (3.12)$$

Proof Integrating on both sides of (3.6) with respect to ξ_1, one gets

$$\int_{1/2}^{\xi_1}\frac{\partial}{\partial\xi_1}(\Lambda_{11})d\xi+\int_{1/2}^{\xi_1}\frac{\partial}{\partial\xi_2}(\Lambda_{21})d\xi_1=0$$

Consequently

$$\Lambda_{11}+\frac{\partial}{\partial\xi_2}\left(\int_{1/2}^{\xi_1}\Lambda_{21}d\xi_1\right)=\phi_2(\xi_2)$$

Set $\tilde{\Lambda}_{21}=\int_{1/2}^{\xi_1}\Lambda_{21}d\xi_1$. One can directly verify that $\tilde{\Lambda}_{21}$ is symmetric and anti-symmetric with respect to the middle hyperplane Δ_1 and Δ_2, respectively, due to Λ_{21} are all anti-symmetric with respect to the middle hyperplanes Δ_1, Δ_2. Hence

$$\tilde{\Lambda}_{21}(\xi_1,\xi_2)\sim\sum(-1)^{m_1+m_2+1}\alpha_{m_1m_2}cos2\pi m_1\xi_1\cdot sin2\pi m_2\xi_2, \qquad (3.13)$$

It follows from Lemma 3.2 that

$$\frac{\partial}{\partial\xi_2}(\tilde{\Lambda}_{21})=-\sum_{m_1,m_2}(-1)^{m_1+m_2}A_{m_1m_2}cos2\pi m_1\xi_1\cdot cos2\pi m_2\xi_2+\phi_2(\xi_2)$$

Integrating on both sides of the above equation with respect to ξ_2, one gets

$$\begin{aligned}\tilde{\Lambda}_{21} = &-\sum_{m_1,m_2} \frac{(-1)^{m_1+m_2}}{2\pi m_2} A_{m_1m_2} cos2\pi m_1\xi_1 \cdot sin2\pi m_2\xi_2 \\ &+ \int_{1/2}^{\xi_2} \phi_2(\xi_2)d\xi_2 + \phi_1(\xi_1)\end{aligned} \tag{3.14}$$

Comparing (3.13) with (3.14), it gives

$$2\pi m_2 \alpha_{m_1m_2} = A_{m_1m_2}, \quad m_1 \neq 0, m_2 \neq 0 \tag{3.15}$$

On the other hand, recalling the definition of the coefficients of Fourier series and integrating by parts, one gets

$$\begin{aligned}\alpha_{m_1m_2} &= \int_0^1 \int_0^1 \tilde{\Lambda}_{21} cos2\pi m_1\xi_1 \cdot sin2\pi m_2\xi_2 d\xi_1 d\xi_2 \\ &= \int_0^1 sin2\pi m_2\xi_2 d\xi_2 \int_0^1 (\int_{1/2}^{\xi_1} \Lambda_{21}(t,\xi_2)dt) cos2\pi m_1\xi_1 d\xi_1 \\ &= -\frac{1}{2\pi m_1} \int_0^1 \int_0^1 \Lambda_{21}(\xi) sin2\pi m_1\xi_1 \cdot sin2\pi m_2\xi_2 d\xi_1 d\xi_2 \\ &= -\frac{1}{2\pi m_1} B_{m_1m_2}\end{aligned} \tag{3.16}$$

Recall that $\Lambda_{11}, \Lambda_{21}$ are symmetric and anti-symmetric with respect to the middle hyperplanes Δ_1, Δ_2, respectively, it implies

$$\Lambda_{11} \sim \sum (-1)^{m_1+m_2} A_{m_1m_2} cos2\pi m_1\xi_1 \cdot cos2\pi m_2\xi_2 \tag{3.17}$$

$$\Lambda_{21} \sim \sum (-1)^{m_1+m_2} B_{m_1m_2} sin2\pi m_1\xi_1 \cdot sin2\pi m_2\xi_2 \tag{3.18}$$

Combining (3.15) with (3.16), this yields

$$m_1 A_{m_1m_2} + m_2 B_{m_1m_2} = 0, \quad m_1 \neq 0, m_2 \neq 0$$

Comparing with the relation of the coefficients between real Fourier's series and complex Fourier's series(see, [19], pp. 190), one can get

$$m_1 c^{(1)}_{m_1m_2} + m_2 d^{(1)}_{m_1m_2} = 0, \quad m_1 \neq 0, m_2 \neq 0 \tag{3.19}$$

Combining (3.7) and (3.19), it leads

$$m_1 c^{(1)}_{m_10} + m_2 d^{(1)}_{0m_2} = 0, \quad m_1 \neq 0, m_2 \neq 0 \tag{3.20}$$

Therefore the proof of Proposition 3.1 is complete. □

Theorem 3.1 *Under the assumptions of Lemma 3.1, it can be proven that* $\sigma_\xi(N_1^*)$, $\sigma_\xi(N_2^*)$ *are continuous on the boundary* ∂Q, *where* $\sigma_\xi(N_1^*)$, $\sigma_\xi(N_2^*)$ *denote the normal derivatives of* $N_1^*(\xi)$ *and* $N_2^*(\xi)$, *respectively.*

Proof Set $v_1(\xi) = e^{i2\pi m_2 \xi_2} - 1$, the variational formulation of (3.2) can be rewritten as

$$\int_{l_1} [\Lambda_{11}] v_1(\xi) d\Gamma + i2\pi m_2 d_{0,m_2}^{(1)} = 0 \tag{3.21}$$

where side l_1 as shown in Fig. 3.2, and $[\Lambda_{11}]$ denotes the jump of the function Λ_{11} on the side l_1 of ∂Q.

Since $\Lambda_{11}|_{l_1} = \Lambda_{11}|_{l_2}$, *i.e.* $[\Lambda_{11}]|_{l_1} = 0$, one gets $d_{0,m_2}^{(1)} = 0$.

Putting it into (3.20), one has $c_{m_1 0}^{(1)} = 0$.

Setting $v_2(\xi) = e^{i2\pi m_1 \xi_1} - 1$,, the variational formulation of (3.2) can similarly be written as:

$$\int_{l_3} [\Lambda_{21}] v_2(\xi) d\Gamma + i2\pi m_1 c_{m_1 0}^{(1)} = 0 \tag{3.22}$$

where side l_3 as shown in Fig. 3.2, and $[\Lambda_{21}]$ denotes the jump of the function Λ_{21} on the side l_3 of ∂Q, and consequently

$$\int_{l_3} [\Lambda_{21}] v_2(\xi) d\Gamma = 0 \tag{3.23}$$

Integrating directly on both sides of (3.2), one has

$$\int_{\partial Q} \sigma_\xi(N_1^*) d\Gamma = 0 \tag{3.24}$$

Observing that $\int_{l_1} [\Lambda_{11}] d\Gamma = 0$, thanks to Lemma 3.1, and using (3.20), one derives $\int_{l_3} [\Lambda_{21}] d\Gamma = 0$.

To summarize the above results, one can deduce that

$$\int_{l_3} [\Lambda_{21}] e^{i2\pi m_1 \xi_1} d\xi_1 = 0, \quad \forall m_1 \in Z$$

The completeness of the function family $\{e^{i2\pi m_1 \xi_1}\}_{m_1=-\infty}^{+\infty}$ implies that $[\Lambda_{21}]|_{l_3} = 0$. Hence we deduce that $\sigma_\xi(N_1^*)$ is continuous on ∂Q.

The remainder can be completed in a similar way.

From Lemma 3.1, one can directly verify that $\sigma_\xi(N_{12}^*)$, $\sigma_\xi(N_{21}^*)$ are continuous on the boundary ∂Q. Only we next consider $\sigma_\xi(N_{11}^*)$, $\sigma_\xi(N_{22}^*)$.

From (3.3), let $\Lambda_{1,11} = (a_{11}^*(\xi) N_1^*(\xi) + \sum_{j=1}^{2} a_{1j}^*(\xi) \frac{\partial N_{11}^*(\xi)}{\partial \xi_j})$,

$$\Lambda_{2,11} = \left(a_{21}^*(\xi) N_1^*(\xi) + \sum_{j=1}^{2} a_{2j}^*(\xi) \frac{\partial N_{11}^*(\xi)}{\partial \xi_j} \right).$$

Similarly to (3.7), one gets

$$\begin{aligned}&-i2\pi m_1(c^{(11)}_{m_1m_2}-c^{(11)}_{m_10})-i2\pi m_2(d^{(11)}_{m_1m_2}-d^{(11)}_{0m_2})\\&=-c^{(1)}_{m_1m_2}+c^{(1)}_{m_10}+c^{(1)}_{0m_2},m_1\neq 0,m_2\neq 0\end{aligned}\qquad(3.25)$$

where $c^{(11)}_{m_1m_2}=\int\limits_Q\Lambda_{1,11}(\xi)e^{i2\pi m\cdot\xi}d\xi,\qquad c^{(11)}_{m_10}=\int\limits_Q\Lambda_{1,11}(\xi)e^{i2\pi m_1\xi_1}d\xi;$

$d^{(11)}_{m_1m_2}=\int\limits_Q\Lambda_{2,11}(\xi)e^{i2\pi m\cdot\xi}d\xi,\qquad d^{(11)}_{0m_2}=\int\limits_{Q\cap\omega}\Lambda_{2,11}(\xi)e^{i2\pi m_2\xi_2}d\xi$, and $c^{(1)}_{m_1m_2}$, $c^{(1)}_{m_10}$, $c^{(1)}_{0m_2}$ are stated in (3.7).

Setting $G(\xi_1,\xi_2)=\frac{\partial}{\partial\xi_2}(\Lambda_{2,11})=\frac{\partial}{\partial\xi_2}\Big[a^*_{21}(\xi)N^*_1(\xi)+\sum\limits_{j=1}^{2}a^*_{2j}(\xi)\frac{\partial N^*_{11}}{\partial\xi_j}\Big]$, and following along the lines of the proof of Proposition 3.1, one has the following proposition:

Proposition 3.2 *Under the assumptions of Lemma 3.1, then we have the following equalities:*

$$-i2\pi m_1c^{(11)}_{m_1m_2}-i2\pi m_2d^{(11)}_{m_1m_2}=-c^{(1)}_{m_1m_2},\quad m_1\neq 0,m_2\neq 0\qquad(3.26)$$

and

$$i2\pi m_1c^{(11)}_{m_10}+i2\pi m_2d^{(11)}_{0m_2}=c^{(1)}_{m_10}+c^{(1)}_{0m_2},\quad m_1\neq 0,m_2\neq 0\qquad(3.27)$$

Theorem 3.2 *Under the assumptions of Lemma 3.1, it can be proven that* $\sigma_\xi(N^*_{\alpha_1\alpha_2})$, $\alpha_j=1,2$, *are continuous on the boundary* ∂Q, *where* $\sigma_\xi(N^*_{\alpha_1\alpha_2})$, $\alpha_j=1,2$ *denote the normal derivatives of* $N^*_{\alpha_1\alpha_2}(\xi)$, *respectively.*

Proof Set $v_1(\xi)=(e^{i2\pi m_2\xi_2}-1)$, $v_2(\xi)=(e^{i2\pi m_1\xi_1}-1)$, then the variational formulations of equation(3.3) are the following:

$$\int\limits_{l_1}[\Lambda_{1,11}(\xi)]v_1(\xi)d\Gamma=i2\pi m_2d^{(11)}_{0m_2}-c^{(1)}_{0m_2};$$

$$\int\limits_{l_3}[\Lambda_{2,11}(\xi)]v_2(\xi)d\Gamma=i2\pi m_1c^{(11)}_{m_10}-c^{(1)}_{m_10}$$

Proposition 3.2 ensures that

$$\begin{aligned}\int\limits_{l_1}[\Lambda_{1,11}(\xi)]v_1(\xi)d\Gamma+\int\limits_{l_3}[\Lambda_{2,11}(\xi)]v_2(\xi)d\Gamma&=i2\pi m_2d^{(11)}_{0m_2}+i2\pi m_1c^{(11)}_{m_10}\\&\quad-c^{(1)}_{0m_2}-c^{(1)}_{m_10}=0\end{aligned}\qquad(3.28)$$

Set $\lambda_{m_2}=\int\limits_{l_1}[\Lambda_{1,11}]e^{i2\pi m_2\xi_2}d\xi_2,\ \mu_{m_1}=\int\limits_{l_3}[\Lambda_{2,11}]e^{i2\pi m_1\xi_1}d\xi_1,\ m_1\neq 0,\ m_2\neq 0;$ $\lambda_0=\int\limits_{l_1}[\Lambda_{1,11}]d\xi_2,\ \mu_0=\int\limits_{l_3}[\Lambda_{2,11}]d\xi_1.$

(3.28) implies that

$$\lambda_{m_2} - \lambda_0 + \mu_{m_1} - \mu_0 = 0 \tag{3.29}$$

Integrating directly on both sides of (3.3), one gets $\lambda_0 + \mu_0 = 0$, and consequently

$$\lambda_{m_2} + \mu_{m_1} = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{3.30}$$

Let $m_2 \to +\infty$, for any fixed m_1, and using Riemann-Lebesgue lemma (see [19]), one derives $\mu_{m_1} = 0$, for $\forall m_1 \neq 0$. Similarly, one has $\lambda_{m_2} = 0$, for $\forall m_2 \neq 0$. Hence one can deduce that $[\Lambda_{1,11}] = const$, $[\Lambda_{2,11}] = const$. Therefore it implies that $\sigma_\xi(N^*_{11})$ has a constant sign in the neighborhood of ∂Q.

The fact $\int\limits_{\partial Q} \sigma_\xi(N^*_{11})d\Gamma = 0$ gives $\sigma_\xi(N^*_{11})|_{\partial Q} = 0$. Similarly, one can prove that $\sigma_\xi(N^*_{22})|_{\partial Q} = 0$.

The proof of Theorem 3.2 is complete. □

Theorem 3.3 *Under the assumptions of Lemma 3.1, we can say that* $\sigma_\xi(N^*_{\alpha_1\alpha_2\alpha_3})$, $\alpha_j = 1, 2$, *are continuous on the boundary* ∂Q, *where* $\sigma_\xi(N^*_{\alpha_1\alpha_2\alpha_3})$ *denote the normal derivatives of* $N^*_{\alpha_1\alpha_2\alpha_3}(\xi)$, $\alpha_j = 1, 2$, *respectively.*

Proof Following along the lines of proofs of Theorem 3.1 and Theorem 3.2, one can easily complete the proof of Theorem 3.3. □

Remark 3.1 It should be emphasized that, generally speaking, $\sigma_\xi(N^*_{1111})$ is not continuous on the boundary ∂Q, due to $\int\limits_{\partial Q} \sigma_\xi(N^*_{1111})d\Gamma = \int\limits_{Q} \left(a^*_{11}\frac{\partial N^*_{111}}{\partial \xi_1} + a^*_{11}N^*_{11}\right) d\xi \neq 0$. Therefore, under the assumptions of Lemma 3.1, at most one can obtain the error estimate with order $O(\varepsilon^3)$ for the multiscale asymptotic methods.

4 Further Results

In this section, we shall give the further theoretical results, i.e. the normal derivatives of cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $N_{\alpha_1\alpha_2\alpha_3}(\xi)$ defined in the unit cell $Q \cap \omega$ are continuous in a sense on ∂Q, i.e.

Theorem 4.1 *Let* $N_{\alpha_1}(\xi), \ldots, N_{\alpha_1\cdots\alpha_l}(\xi)$ *be the weak solution of problem (2.2), (2.3) and (2.5), respectively. Under the assumptions* $(A_1)-(A_3)$, $(B_1)-(B_3)$, (HS), (MS), *and* $a_{ij}(\xi) \in C^0(\overline{Q \cap \omega})$. *For* $\forall v \in H^1_{per}(Q \cap \omega)$, *then it holds*

$$\left|\int\limits_{\partial Q} \nu_i a_{ij}\frac{\partial N_{\alpha_1}}{\partial \xi_j} v(\xi)d\Gamma_\xi\right| \leq C \cdot \delta^{\frac{\beta_0}{2}} \tag{4.1}$$

$$\left|\int\limits_{\partial Q} \nu_i a_{ij}\frac{\partial N_{\alpha_1\cdots\alpha_l}}{\partial \xi_j} v(\xi)d\Gamma_\xi\right| \leq C \cdot \delta^{\frac{\beta_0}{2}}, \quad l = 2, 3 \tag{4.2}$$

where C is a constant independent of ε.

By Theorems 3.1, 3.2 and 3.3 in the previous section, it only remains necessary to compare with the difference between a set of functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $N_{\alpha_1\alpha_2\alpha_3}(\xi)$ and the corresponding ones $N^*_{\alpha_1}(\xi)$, $N^*_{\alpha_1\alpha_2}(\xi)$, $N^*_{\alpha_1\alpha_2\alpha_3}(\xi)$ in the unit cell $Q\cap\omega$.

To start with, following along the lines of proof of Lemma 4.1 of [18, pp. 45]), one can extend $N_{\alpha_1}(\xi)$ in the unit cell $Q\cap\omega$ as the function $\tilde{N}_{\alpha_1}(\xi)$ in the unit cube Q such that $\tilde{N}_{\alpha_1}\in H^1(Q)$, and $\|\tilde{N}_{\alpha_1}\|_{1,Q}\le C\|N_{\alpha_1}\|_{1,Q\cap\omega}$.

Recalling condition (B_3), and setting $w(\xi)=N^*_{\alpha_1}-\tilde{N}_{\alpha_1}\in H^1_0(Q)$, then we have

$$\int\limits_Q a^*_{ij}\frac{\partial N^*_{\alpha_1}}{\partial\xi_j}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi=-\int\limits_Q a^*_{i\alpha_1}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi \tag{4.3}$$

and the variational form of (2.2) is the following:

$$\int\limits_{Q\cap\omega} a_{ij}\frac{\partial N_{\alpha_1}}{\partial\xi_j}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi=-\int\limits_{Q\cap\omega} a_{i\alpha_1}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi \tag{4.4}$$

Subtracting (4.3) from (4.4), it gives

$$\begin{aligned}0&\le\int\limits_{Q\cap\omega} a_{ij}\frac{\partial(N^*_{\alpha_1}-N_{\alpha_1})}{\partial\xi_j}\frac{\partial(N^*_{\alpha_1}-N_{\alpha_1})}{\partial\xi_i}d\xi\\&=-\int\limits_{Q\backslash\bar{\omega}}\left(a^*_{i\alpha_1}+a^*_{ij}\frac{\partial\tilde{N}_{\alpha_1}}{\partial\xi_j}\right)\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi\\&\quad+\int\limits_{Q\backslash\bar{\omega}} a^*_{ij}\frac{\partial(\tilde{N}_{\alpha_1}-N^*_{\alpha_1})}{\partial\xi_j}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi\end{aligned} \tag{4.5}$$

i.e.

$$\begin{aligned}&-\int\limits_{Q\backslash\bar{\omega}}\left(a^*_{i\alpha_1}+a^*_{ij}\frac{\partial\tilde{N}_{\alpha_1}}{\partial\xi_j}\right)\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi\\&\quad\ge\int\limits_{Q\backslash\bar{\omega}} a^*_{ij}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_j}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi\ge0\end{aligned} \tag{4.6}$$

Hence

$$\begin{aligned}\delta^{1/8}\|N^*_{\alpha_1}-\tilde{N}_{\alpha_1}\|^2_{1,Q\backslash\bar{\omega}}&\le\int\limits_{Q\backslash\bar{\omega}} a^*_{ij}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_j}\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi\\&\le-\int\limits_{Q\backslash\bar{\omega}}\left(a^*_{i\alpha_1}+a^*_{ij}\frac{\partial\tilde{N}_{\alpha_1}}{\partial\xi_j}\right)\frac{\partial(N^*_{\alpha_1}-\tilde{N}_{\alpha_1})}{\partial\xi_i}d\xi\\&\le C\cdot\|N^*_{\alpha_1}-\tilde{N}_{\alpha_1}\|_{1,Q\backslash\bar{\omega}}\end{aligned} \tag{4.7a}$$

i.e.

$$\|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1,Q\backslash\bar{\omega}} \le C \cdot \delta^{-1/8} \tag{4.7b}$$

In particular

$$\|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1,\mathcal{V}_\delta} \le C \cdot \delta^{-1/8} \tag{4.7c}$$

Let

$$I_1^\delta = -\int\limits_{\mathcal{V}_\delta} a^*_{i\alpha_1} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi, \quad I_1^0 = -\int\limits_{\mathcal{V}_0} a^*_{i\alpha_1} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi$$

$$I_2^\delta = -\int\limits_{\mathcal{V}_\delta} a^*_{ij} \frac{\partial \tilde{N}_{\alpha_1}}{\partial \xi_j} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi, \quad I_2^0 = -\int\limits_{\mathcal{V}_0} a^*_{ij} \frac{\partial \tilde{N}_{\alpha_1}}{\partial \xi_j} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi$$

$$I_3^\delta = -\int\limits_{\mathcal{V}_\delta} a^*_{ij} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_j} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi \le 0$$

$$I_3^0 = -\int\limits_{\mathcal{V}_0} a^*_{ij} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_j} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi \le 0$$

$$\begin{aligned} |I_1^\delta| &= |\int\limits_{\mathcal{V}_\delta} a^*_{i\alpha_1} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi| \le C \int\limits_{\mathcal{V}_\delta} |\nabla(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})| d\xi \\ &\le C\|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1,\mathcal{V}_\delta} \cdot \Big(meas(\mathcal{V}_\delta)\Big)^{1/2} \\ &\le C \cdot \delta^{1/2} \|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1,\mathcal{V}_\delta} \\ &\le C_1 \cdot \delta^{1/2} \cdot \delta^{-1/8} = C_1 \cdot \delta^{3/8} \end{aligned} \tag{4.8}$$

It follows from Lemma 3.12 and Remark 3.15 of (see, [15], pp.112-116) that

$$\|\nabla \tilde{N}_{\alpha_1}\|_{0,\gamma,\mathcal{V}_\delta} = \Big(\int\limits_{\mathcal{V}_\delta} |\nabla \tilde{N}_{\alpha_1}|^\gamma d\xi\Big)^{1/\gamma} \le C_0 \Big(\int\limits_{Q\cap\omega} |\nabla N_{\alpha_1}|^s d\xi\Big)^{1/s} \le M \tag{4.9}$$

where $1 < \gamma < \gamma_0 = \dfrac{4s}{4+s}, \quad 4/3 \le s \le +\infty.$

$$\begin{aligned} |I_2^\delta| &= |\int\limits_{\mathcal{V}_\delta} a^*_{ij} \frac{\partial \tilde{N}_{\alpha_1}}{\partial \xi_j} \frac{\partial(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi| \\ &\le C \cdot \Big(\int\limits_{\mathcal{V}_\delta} |\nabla \tilde{N}_{\alpha_1}|^{\gamma_0} d\xi\Big)^{1/\gamma_0} \cdot \Big(\int\limits_{\mathcal{V}_\delta} |\nabla(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})|^{\frac{\gamma_0}{\gamma_0-1}} d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}} \\ &\le C \cdot \|N_{\alpha_1}\|_{1,\infty,Q\cap\omega} \cdot \Big(\int\limits_{\mathcal{V}_\delta} |\nabla(N^*_{\alpha_1} - \tilde{N}_{\alpha_1})|^{\frac{\gamma_0}{\gamma_0-1}} d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}} \end{aligned}$$

$$
\begin{aligned}
&\leq C \cdot \Big(\int\limits_{\mathcal{V}_\delta} |\nabla (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})|^{\frac{\gamma_0}{\gamma_0-1}} d\xi \Big)^{\frac{\gamma_0-1}{\gamma_0}} \\
&\leq C_2 \cdot \Big[\big(\int\limits_{\mathcal{V}_\delta} 1 \cdot d\xi \big)^{\frac{\gamma_0-2}{2(\gamma_0-1)}} \cdot \Big(\int\limits_{\mathcal{V}_\delta} |\nabla (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})|^2 d\xi \Big)^{\frac{\gamma_0}{2(\gamma_0-1)}} \Big]^{\frac{\gamma_0-1}{\gamma_0}} \\
&= C_2 \cdot \Big[\delta^{\frac{\gamma_0-2}{2(\gamma_0-1)}} \cdot \delta^{-\frac{\gamma_0}{8(\gamma_0-1)}} \Big]^{\frac{\gamma_0-1}{\gamma_0}} = C_2 \cdot \delta^{\frac{3\gamma_0-8}{8\gamma_0}} = C_2 \cdot \delta^{\beta_0} \qquad (4.10)
\end{aligned}
$$

where $0 < \beta_0 < 1/8$, if γ_0 is close to 4, then we have

$$
|I_1^\delta| + |I_2^\delta| \leq C \cdot \delta^{min(3/8, \beta_0)} \leq C \cdot \delta^{\beta_0} \qquad (4.11)
$$

From (4.5) and $I_3^\delta \leq 0$, we have

$$
0 \leq \int\limits_{Q \cap \omega} a_{ij} \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_j} \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi \leq I_1^\delta + I_2^\delta + I_1^0 + I_2^0 + I_3^0 \qquad (4.12)
$$

If $|I_1^0 + I_2^0 + I_3^0| \leq (C_1 + C_2)\delta^{\beta_0}$, from (4.12), then it leads

$$
\|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1, Q \cap \omega} \leq C \cdot \delta^{\frac{\beta_0}{2}} \qquad (4.13)
$$

Otherwise, if $|I_1^0 + I_2^0 + I_3^0| > (C_1 + C_2)\delta^{\beta_0} > 0$, from (4.5), one has $I_1^0 + I_2^0 + I_3^0 > 0$, and consequently

$$
\begin{aligned}
\delta^{1/8} \|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|^2_{1, \mathcal{V}_0} &\leq \int\limits_{\mathcal{V}_0} a^*_{ij} \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_j} \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi \\
&\leq - \int\limits_{\mathcal{V}_0} (a^*_{i\alpha_1} + a^*_{ij} \frac{\partial \tilde{N}_{\alpha_1}}{\partial \xi_j}) \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi \\
&\leq C \cdot \delta^{1/8} \|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1, \mathcal{V}_0} \qquad (4.14a)
\end{aligned}
$$

i.e.

$$
\|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1, \mathcal{V}_0} \leq C \qquad (4.14b)
$$

Hence

$$
\begin{aligned}
|I_1^0| = | \int\limits_{\mathcal{V}_0} a^*_{i\alpha_1} \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi | &\leq \delta^{1/8} \int\limits_{\mathcal{V}_0} |\nabla (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})| d\xi \\
\leq C \cdot \delta^{1/8} \|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|_{1, \mathcal{V}_0} \leq C \cdot \delta^{1/8} & \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
|I_2^0| &= | \int\limits_{\mathcal{V}_0} a^*_{ij} \frac{\partial (\tilde{N}_{\alpha_1} - N^*_{\alpha_1})}{\partial \xi_j} \frac{\partial (N^*_{\alpha_1} - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi | \\
&\leq C \cdot \delta^{1/8} \|N^*_{\alpha_1} - \tilde{N}_{\alpha_1}\|^2_{1, \mathcal{V}_0} \leq C \cdot \delta^{1/8} \qquad (4.16)
\end{aligned}
$$

$$|I_3^0| = |\int\limits_{\mathcal{V}_0} a_{ij}^* \frac{\partial \tilde{N}_{\alpha_1}}{\partial \xi_j} \frac{\partial (N_{\alpha_1}^* - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi| \leq C \cdot \delta^{1/8} \|N_{\alpha_1}^* - \tilde{N}_{\alpha_1}\|_{1,\mathcal{V}_0} \leq C \cdot \delta^{1/8} \tag{4.17}$$

and

$$|\int\limits_{Q\cap\omega} a_{ij} \frac{\partial (N_{\alpha_1}^* - \tilde{N}_{\alpha_1})}{\partial \xi_j} \frac{\partial (N_{\alpha_1}^* - \tilde{N}_{\alpha_1})}{\partial \xi_i} d\xi| \leq (C_1 + C_2)\delta^{\beta_0} + I_1^0 + I_2^0 + I_3^0 \leq C \cdot \delta^{min(1/8,\beta_0)} = C \cdot \delta^{\beta_0} \tag{4.18}$$

Therefore, we have

$$\|N_{\alpha_1}^* - \tilde{N}_{\alpha_1}\|_{1,Q\cap\omega} \leq C \cdot \delta^{\frac{\beta_0}{2}}, 0 < \beta_0 < 1/8 \tag{4.19}$$

Let us turn to the proof of Theorem 4.1.

Proof For $v \in H_{per}^1(Q \cap \omega)$, following along the lines of proof of Lemma 4.1 of [18, pp.45]), one can extend it as the function $\tilde{v} \in H^1(Q)$ such that $\|\tilde{v}\|_{1,Q} \leq C\|v\|_{1,Q\cap\omega}$.

From (3.2) and (2.2), we have

$$-\int\limits_{\partial Q} \nu_i a_{ij}^* \frac{\partial N_{\alpha_1}^*}{\partial \xi_j} v(\xi) d\Gamma_\xi + \int\limits_Q a_{ij}^* \frac{\partial N_{\alpha_1}^*}{\partial \xi_j} \frac{\partial \tilde{v}}{\partial \xi_i} d\xi = -\int\limits_Q a_{i\alpha_1}^* \frac{\partial \tilde{v}}{\partial \xi_i} d\xi \tag{4.20}$$

$$-\int\limits_{\partial Q} \nu_i a_{ij} \frac{\partial N_{\alpha_1}}{\partial \xi_j} v(\xi) d\Gamma_\xi + \int\limits_{Q\cap\omega} a_{ij} \frac{\partial N_{\alpha_1}}{\partial \xi_j} \frac{\partial v}{\partial \xi_i} d\xi = -\int\limits_{Q\cap\omega} a_{i\alpha_1} \frac{\partial v}{\partial \xi_i} d\xi \tag{4.21}$$

Recalling Theorem 3.1, and subtracting (4.20) from (4.21), one gets

$$\int\limits_{\partial Q} \nu_i a_{ij} \frac{\partial N_{\alpha_1}}{\partial \xi_j} v(\xi) d\Gamma_\xi = \int\limits_{Q\cap\omega} a_{ij} \frac{\partial (N_{\alpha_1} - N_{\alpha_1}^*)}{\partial \xi_j} \frac{\partial v}{\partial \xi_i} d\xi - \int\limits_{Q\backslash\bar{\omega}} a_{ij}^* \frac{\partial N_{\alpha_1}^*}{\partial \xi_j} \frac{\partial \tilde{v}}{\partial \xi_i} d\xi - \int\limits_{Q\backslash\bar{\omega}} a_{i\alpha_1}^* \frac{\partial \tilde{v}}{\partial \xi_i} d\xi \tag{4.22}$$

$$|\int\limits_{Q\cap\omega} a_{ij} \frac{\partial (N_{\alpha_1} - N_{\alpha_1}^*)}{\partial \xi_j} \frac{\partial v}{\partial \xi_i} d\xi| \leq C \cdot \|N_{\alpha_1} - N_{\alpha_1}^*\|_{1,Q\cap\omega} \|v\|_{1,Q\cap\omega} \leq C \cdot \delta^{\frac{\beta_0}{2}} \|v\|_{1,Q\cap\omega} \tag{4.23}$$

$$\begin{aligned}
&|\int_{\mathcal{V}_\delta} a^*_{ij}\frac{\partial(N_{\alpha_1}-N^*_{\alpha_1})}{\partial\xi_j}\frac{\partial\tilde{v}}{\partial\xi_i}d\xi|\\
&\quad\le C\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla(N_{\alpha_1}-N^*_{\alpha_1})|^{\frac{\gamma_0}{\gamma_0-1}}d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}}\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla\tilde{v}|^{\gamma_0}d\xi\Big)^{\frac{1}{\gamma_0}}\\
&\quad\le C\cdot\|v\|_{1,\infty,\mathcal{V}_\delta}\Big(\int_{\mathcal{V}_\delta}|\nabla(N_{\alpha_1}-N^*_{\alpha_1})|^{\frac{\gamma_0}{\gamma_0-1}}d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}}\\
&\quad\le C\cdot\Big[\Big(\int_{\mathcal{V}_\delta}1\cdot d\xi\Big)^{\frac{\gamma_0-2}{2(\gamma_0-1)}}\int_{\mathcal{V}_\delta}|\nabla(N_{\alpha_1}-N^*_{\alpha_1})|^2d\xi\Big)^{\frac{\gamma_0}{2(\gamma_0-1)}}\Big]^{\frac{\gamma_0-1}{\gamma_0}}\\
&\quad\le C\cdot\delta^{\frac{3\gamma_0-8}{8\gamma_0}}=C\cdot\delta^{\beta_0}
\end{aligned}\tag{4.24a}$$

$$\begin{aligned}
|\int_{\mathcal{V}_0} a^*_{ij}\frac{\partial(\tilde{N}_{\alpha_1}-N^*_{\alpha_1})}{\partial\xi_j}\frac{\partial\tilde{v}}{\partial\xi_i}d\xi| &\le C\cdot\delta^{1/8}\|\tilde{N}_{\alpha_1}-N^*_{\alpha_1}\|_{1,\mathcal{V}_0}\|\tilde{v}\|_{1,\mathcal{V}_0}\\
&\le C\cdot\delta^{1/8}\|v\|_{1,Q\cap\omega}
\end{aligned}\tag{4.24b}$$

$$\begin{aligned}
|\int_{\mathcal{V}_0} a^*_{ij}\frac{\partial\tilde{N}_{\alpha_1}}{\partial\xi_j}\frac{\partial\tilde{v}}{\partial\xi_i}d\xi| &\le C\cdot\delta^{1/8}\|\tilde{N}_{\alpha_1}\|_{1,\mathcal{V}_0}\|\tilde{v}\|_{1,\mathcal{V}_0}\\
&\le C\cdot\delta^{1/8}\|N_{\alpha_1}\|_{1,Q\cap\omega}\|v\|_{1,Q\cap\omega}
\end{aligned}\tag{4.24c}$$

$$\begin{aligned}
|\int_{\mathcal{V}_\delta} a^*_{ij}\frac{\partial\tilde{N}_{\alpha_1}}{\partial\xi_j}\frac{\partial\tilde{v}}{\partial\xi_i}d\xi| &\le C\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla\tilde{N}_{\alpha_1}|^{\frac{\gamma_0}{\gamma_0-1}}d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}}\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla\tilde{v}|^{\gamma_0}d\xi\Big)^{\frac{1}{\gamma_0}}\\
&\le C\cdot\|v\|_{1,\infty,Q\cap\omega}\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla\tilde{N}_{\alpha_1}|^{\frac{\gamma_0}{\gamma_0-1}}d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}}\\
&\le C\cdot\Big[\Big(\int_{\mathcal{V}_\delta}1\cdot d\xi\Big)^{\frac{\gamma_0-2}{2(\gamma_0-1)}}\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla\tilde{N}_{\alpha_1}|^2d\xi\Big)^{\frac{\gamma_0}{2(\gamma_0-1)}}\Big]^{\frac{\gamma_0-1}{\gamma_0}}\\
&=C\cdot\delta^{\frac{\gamma_0-2}{2\gamma_0}}
\end{aligned}\tag{4.24d}$$

$$|\int_{\mathcal{V}_\delta} a^*_{i\alpha_1}\frac{\partial\tilde{v}}{\partial\xi_i}d\xi|\le C\cdot\Big(\int_{\mathcal{V}_\delta}1\cdot d\xi\Big)^{\frac{\gamma_0-1}{\gamma_0}}\cdot\Big(\int_{\mathcal{V}_\delta}|\nabla\tilde{v}|^{\gamma_0}d\xi\Big)^{\frac{1}{\gamma_0}}\le C\cdot\delta^{\frac{\gamma_0-1}{\gamma_0}}\tag{4.24e}$$

$$|\int_{\mathcal{V}_0} a^*_{i\alpha_1}\frac{\partial\tilde{v}}{\partial\xi_i}d\xi|\le C\cdot\delta^{1/8}\|\tilde{v}\|_{1,\mathcal{V}_0}\le C\cdot\|v\|_{1,\infty,Q\cap\omega}\cdot\delta^{1/8}\tag{4.24f}$$

(4.1) follows from (4.22),(4.23) and (4.24a)–(4.24f), and (4.2) will be shown at the end of this section.

We next prove that the two kinds of homogenization methods are equivalent under the assumptions of (HS),and (MS).

By recalling the definitions of cell functions in the unit cell $Q \cap \omega$ in some classical homogenization books(see, e.g. [4,16,18]), one has:

$$z \begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial \breve{N}_{\alpha_1}}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)) \quad \text{in} \quad Q \cap \omega \\ \sigma_\xi(\breve{N}_{\alpha_1}\Big) = -\nu_i a_{i\alpha_1}(\xi), \quad \text{on} \quad Q \cap \partial\omega \\ \breve{N}_{\alpha_1}(\xi) \quad \text{is 1-periodic in } \xi, \quad \int\limits_{Q\cap\omega} \breve{N}_{\alpha_1}(\xi)d\xi = 0 \end{cases} \tag{4.25}$$

$$\breve{\hat{a}}_{ij} = \frac{1}{|Q \cap \omega|} \int\limits_{Q\cap\omega} \Big(a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial \breve{N}_j(\xi)}{\partial \xi_k}\Big)d\xi \tag{4.26}$$

For $v \in H^1_{per}(Q \cap \omega)$, the variational form of (4.25) is the following:

$$\int\limits_{Q\cap\omega} a_{ij}(\xi)\frac{\partial \breve{N}_{\alpha_1}}{\partial \xi_j}\frac{\partial v}{\partial \xi_i}d\xi = -\int\limits_{Q\cap\omega} a_{i\alpha_1}\frac{\partial v}{\partial \xi_i}d\xi \tag{4.27}$$

and the variational form of (2.2) is the following:

$$\int\limits_{\partial Q} \sigma_\xi(N_{\alpha_1})v(\xi)d\Gamma_\xi - \int\limits_{Q\cap\omega} a_{ij}(\xi)\frac{\partial N_{\alpha_1}}{\partial \xi_j}\frac{\partial v}{\partial \xi_i}d\xi = \int\limits_{Q\cap\omega} a_{i\alpha_1}\frac{\partial v}{\partial \xi_i}d\xi \tag{4.28}$$

Subtracting (4.28) from (4.27), this yields

$$\int\limits_{\partial Q} \sigma_\xi(N_{\alpha_1})v(\xi)d\Gamma_\xi = \int\limits_{Q\cap\omega} a_{ij}(\xi)\frac{\partial (N_{\alpha_1} - \breve{N}_{\alpha_1})}{\partial \xi_j}\frac{\partial v}{\partial \xi_i}d\xi \tag{4.29}$$

Set $v = N_{\alpha_1} - \breve{N}_{\alpha_1}$, it follows from (4.1) and (4.27) that

$$\|N_{\alpha_1} - \breve{N}_{\alpha_1}\|_{1,Q\cap\omega} \le C \cdot \delta^{\frac{\beta_0}{2}} \tag{4.30}$$

and consequently

$$|\breve{\hat{a}}_{ij} - \hat{a}_{ij}| \le \frac{1}{|Q \cap \omega|} \int\limits_{Q\cap\omega} |a_{ik}(\xi)\frac{\partial (\breve{N}_j - N_j)}{\partial \xi_k}|d\xi \le C \cdot \delta^{\frac{\beta_0}{2}} \tag{4.31}$$

As $\delta \to 0$, we have $\breve{\hat{a}}_{ij} = \hat{a}_{ij}$. In addition, it is obvious that $\int_{Q\cap\omega} N_{\alpha_1}(\xi)d\xi = 0$, thanks to Lemma 3.1. Therefore, we can obtain the following theorem:

Theorem 4.2 *It can be proven that*

$$\breve{\hat{a}}_{ij} = \hat{a}_{ij}, \quad N_{\alpha_1} = \breve{N}_{\alpha_1}, \quad \alpha_1 = 1, 2, \ldots, n \tag{4.32}$$

Recall the definitions of $\breve{N}_{\alpha_1\alpha_1}$ *in the unit cell* $Q \cap \omega$ *in some books (see, [18], pp.125, (1.17)):*

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial \breve{N}_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)\breve{N}_{\alpha_2}(\xi)\Big) \\ -a_{\alpha_1 j}(\xi)\dfrac{\partial \breve{N}_{\alpha_2}(\xi)}{\partial \xi_j} - a_{\alpha_1\alpha_2}(\xi) + \breve{\hat{a}}_{\alpha_1\alpha_2}, \quad in \quad Q \cap \omega \\ \sigma_\xi(\breve{N}_{\alpha_1\alpha_2}) = -\nu_i a_{i\alpha_1}(\xi)\breve{N}_{\alpha_2}(\xi), \quad on \quad Q \cap \partial\omega \\ \breve{N}_{\alpha_1\alpha_2}(\xi) \text{is 1-periodic in} \quad \xi, \quad \int\limits_{Q\cap\omega} \breve{N}_{\alpha_1\alpha_2}(\xi)d\xi = 0 \end{cases} \tag{4.33}$$

From (2.3), (4.33) and (4.32), one can deduce that $\breve{N}_{\alpha_1\alpha_2}(\xi) = N_{\alpha_1\alpha_2}(\xi) + C$*, where* C *is any constants. Therefore,* $\sigma(N_{\alpha_1\alpha_2})$ *are continuous on the boundary* ∂Q *of the unit cell* $Q \cap \omega$*. It is a straightforward consequence of the fact that* $\sigma(\breve{N}_{\alpha_1\alpha_2})$ *are continuous on the boundary* ∂Q *of the unit cell* $Q \cap \omega$*.*

Following along the lines of the proof of (4.1) in Theorem 4.1, and using Theorem 3.2 and Theorem 3.3, one has

$$\|N^*_{\alpha_1\cdots\alpha_l} - N_{\alpha_1\cdots\alpha_l}\|_{1,Q\cap\omega} \le c \cdot \delta^{\frac{\beta_0}{2}}, \quad l \ge 2 \tag{4.34}$$

Furthermore

$$|\int\limits_{\partial Q} \nu_i a_{ij}\frac{\partial N_{\alpha_1\cdots\alpha_l}}{\partial \xi_j} v(\xi)d\Gamma_\xi| \le C \cdot \delta^{\frac{\beta_0}{2}}, \quad l = 2, 3 \tag{4.35}$$

Therefore,(4.2) in Theorem 4.1 has been proven.

5 Main Convergence Theorems

In this section, we shall give some main convergence results and their rigorous proofs in this paper.

Theorem 5.1 *Suppose* $\Omega^\varepsilon \subset R^n$ *is the union of entire cells satisfying the conditions* $(B_1) - (B_3)$, (HS), *i.e.* $\overline{\Omega}^\varepsilon = \cup_{z\in T_\varepsilon}\varepsilon(z + \overline{Q \cap \omega})$*. Let* $u^\varepsilon(x)$ *be the weak solution of problem (1.1) with the mixed boundary conditions, and* $u^\varepsilon_s(x)$ *be the approximate solution given in (2.7). Under the assumptions* $(A_1) - (A_3)$*, and* (MS)*, if* $a^\varepsilon_{ij} \in C^0(\bar{\Omega}^\varepsilon)$, $f \in H^s(\Omega)$, $\bar{u} \in H^{s+3/2}(\partial\Omega)$, $u^0 \in H^{s+2}(\Omega)$*, then it holds:*

$$\|u^\varepsilon - u^\varepsilon_s\|_{1,\Omega^\varepsilon} \le \begin{cases} C \cdot \varepsilon^{1/2}, & if \quad s = 1; \\ C \cdot \varepsilon^{min(3,s-1)}, & if \quad s \ge 2 \end{cases} \tag{5.1}$$

where C *is a constant independent of* ε*.*

Proof Assume that Ω^ε is the union of entire periodic cells, i.e. $\overline{\Omega}^\varepsilon = \cup_{z\in T_\varepsilon}\varepsilon(z+\overline{Q\cap\omega})$. Set $\mathcal{L}_\varepsilon u^\varepsilon \equiv -\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon}{\partial x_j}\Big)$.

Taking into account the fact $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \varepsilon^{-1}\frac{\partial}{\partial \xi_i}$, $\xi = \varepsilon^{-1}x$, we have

$$\begin{aligned}
\frac{\partial u_s^\varepsilon}{\partial x_j} &= (\frac{\partial}{\partial x_i} + \varepsilon^{-1}\frac{\partial}{\partial \xi_i})\sum_{l=0}^{s}\varepsilon^l\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi)D^\alpha u^0(x)\\
&= \sum_{l=0}^{s}\varepsilon^l\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi)D^\alpha\frac{\partial u^0(x)}{\partial x_j}\\
&= \sum_{l=0}^{s}\varepsilon^{l-1}\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} \frac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_j}D^\alpha u^0(x)
\end{aligned}$$

Furthermore, we have the following equation which holds in the sense of distributions:

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_s^\varepsilon}{\partial x_j}\Big) &= \sum_{l=0}^{s}\varepsilon^l\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} a_{ij}(\xi)N_{\alpha_1\cdots\alpha_l}(\xi)D^\alpha\frac{\partial^2 u^0(x)}{\partial x_i\partial x_j}\\
&+\sum_{l=0}^{s}\varepsilon^{l-1}\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} a_{ij}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_j}D^\alpha\frac{\partial u^0(x)}{\partial x_i}\\
&+\sum_{l=0}^{s}\varepsilon^{l-1}\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} \frac{\partial(a_{ij}(\xi)N_{\alpha_1\cdots\alpha_l}(\xi))}{\partial \xi_i}D^\alpha\frac{\partial u^0(x)}{\partial x_j}\\
&+\sum_{l=0}^{s}\varepsilon^{l-2}\sum_{\alpha_1,\ldots,\alpha_l=1}^{n} \frac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_j}\Big)D^\alpha u^0(x)\\
&+R_c(\varepsilon,\xi,x)
\end{aligned}$$

From (4.1) and (4.2), we have $\|R_c\|_{0,\partial E_z} \le C\cdot\delta^{\beta_0/2}$, where $E_z = \varepsilon(z+Q)$.

From (2.2),(2.3) and (2.5), if $u^0\in H^{s+2}(\Omega)$, $s\ge 2$, then we have the following equality which holds in the sense of distributions:

$$\begin{aligned}
\mathcal{L}_\varepsilon(u^\varepsilon - u_s^\varepsilon) &= f(x) - \mathcal{L}_\varepsilon u_s^\varepsilon\\
&= \varepsilon^{s-1}\sum_{\alpha_1,\ldots,\alpha_{s+1}=1}^{n} a_{i\alpha_1}(\xi)\frac{\partial N_{\alpha_2\cdots\alpha_{s+1}}(\xi)}{\partial \xi_i}D^\alpha u^0(x)\\
&+\varepsilon^{s-1}\sum_{\alpha_1,\ldots,\alpha_{s+1}=1}^{n}\frac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_{s+1}}(\xi)\Big)D^\alpha u^0(x)\\
&+\varepsilon^{s-1}\sum_{\alpha_1,\ldots,\alpha_{s+1}=1}^{n} a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\ldots\alpha_{s+1}}(\xi)D^\alpha u^0(x)
\end{aligned}$$

$$\begin{aligned}&+\varepsilon^s \sum_{\alpha_1,\dots,\alpha_{s+2}=1}^{n} a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_{s+2}}(\xi)D^\alpha u^0(x)\\&+R_c(\varepsilon,\xi,x)\\=\ &F_0(\varepsilon,\xi,x)+R_c(\varepsilon,\xi,x)\end{aligned}\tag{5.2}$$

where $\|F_0\|_{0,\Omega} \leq C \cdot \varepsilon^{s-1}$.

It is not difficult to check that

$$\begin{aligned}\sigma_\varepsilon(u_s^\varepsilon) &\equiv -\nu_i a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_s^\varepsilon}{\partial x_j}\\&=\sum_{l=1}^{s}\varepsilon^{l-1}\Big(\sigma_\xi(N_{\alpha_1\cdots\alpha_l})+\nu_i a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_l}(\xi)\Big)D^\alpha u^0(x)\\&\quad+O(\varepsilon^s)\end{aligned}\tag{5.3}$$

Suppose $A_{\Omega^\varepsilon}(u,v)=\int\limits_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon})\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx, \quad E_z=\varepsilon(z+Q)$, we have

$$\begin{aligned}A_{\Omega^\varepsilon}\Big(u^\varepsilon-u_s^\varepsilon,u^\varepsilon-u_s^\varepsilon\Big) &= \Big(\mathcal{L}_\varepsilon(u^\varepsilon-u_s^\varepsilon),u^\varepsilon-u_s^\varepsilon\Big)\\&\quad+\sum_{z\in T_\varepsilon}\int\limits_{\partial E_z}\sigma_\varepsilon(u^\varepsilon-u_s^\varepsilon)\cdot(u^\varepsilon-u_s^\varepsilon)d\Gamma_x\\&=(F_0,u^\varepsilon-u_s^\varepsilon)\\&\quad-\sum_{z\in T_\varepsilon}\int\limits_{\partial E_z}\sigma_\varepsilon(u_s^\varepsilon)\cdot(u^\varepsilon-u_s^\varepsilon)d\Gamma_x=(F_0,u^\varepsilon-u_s^\varepsilon)\\&\quad-\sum_{z\in T_\varepsilon}\sum_{l=1}^{s}\int\limits_{\partial E_z}\sigma_\xi(N_{\alpha_1\cdots\alpha_l})D^\alpha u^0(x)\cdot(u^\varepsilon-u_s^\varepsilon)d\Gamma_x\\&\quad-\sum_{z\in T_\varepsilon}\int\limits_{\partial E_z}R_s\cdot(u^\varepsilon-u_s^\varepsilon)d\Gamma_x\end{aligned}\tag{5.4}$$

where $\|R_s\|_{0,\partial E_z} \leq C\cdot\varepsilon^s$.

From (4.1), (4.2),(5.2) and (5.3), one gets

$$\begin{aligned}\|u^\varepsilon-u_s^\varepsilon\|_{1,\Omega^\varepsilon} &\leq C\{\cdot\varepsilon^{min(3,s-1)}+\delta^{\beta_0/2}\}\\&\leq C\cdot\varepsilon^{min(3,s-1)},\quad s\geq 2\end{aligned}\tag{5.5}$$

Here choose a sufficiently small $\delta > 0$ such that $\delta^{\beta_0/2} \leq \varepsilon^{min(3,s-1)}$.

On the other hand, for $s=1$, if $u^0 \in H^3(\Omega)$, by means of Theorem 2.29 of [7, pp. 45](Also, see [18, pp.124]), one has

$$\|u^\varepsilon-u_s^\varepsilon\|_{1,\Omega^\varepsilon}\leq C\cdot\varepsilon^{1/2}\tag{5.6}$$

Therefore the proof of Theorem 5.1 is complete.

Theorem 5.2 *Suppose that Ω^ε is a bounded perforated domain satisfying conditions $(B_1)-(B_3), (HS)$. Let $u^\varepsilon(x)$ be the weak solution of problem (1.1), and $u_s^\varepsilon(x), u^{\varepsilon,b}(x), U_s^\varepsilon(x)$ be given in (2.7), (2.8) and (2.9), respectively. Under the assumption $(A_1)-(A_3), (MS)$, if $a_{ij}^\varepsilon \in C^0(\overline{\Omega}^\varepsilon)$, $f \in H^s(\Omega), \bar{u} \in H^{1/2}(\partial\Omega)$, then it holds*

$$\|u^\varepsilon - U_s^\varepsilon\|_{1,\Omega^\varepsilon} \leq \begin{cases} C\cdot\varepsilon^{1/2}, & s=1 \\ C\cdot min\{\varepsilon^{1/2}, \zeta_s\}, & s\geq 2 \end{cases} \tag{5.7}$$

where $\zeta_s = \varepsilon^{min(3,s-1)} + \Big(\int\limits_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(U_s^\varepsilon)]\Big)^2 d\Gamma\Big)^{1/2}, \quad s\geq 2.$

Proof Under the assumption of (HS) and (MS), using Theorem 4.1 in the previous section, and Theorem 2.29 of [7, pp. 45] (Also, see [18, pp.124]), we have

$$\|u^\varepsilon - u_1^\varepsilon\|_{1,\Omega_0^\varepsilon} \leq C\cdot\varepsilon^{1/2} \tag{5.8}$$

where $\Omega_0^\varepsilon \subset\subset \Omega^\varepsilon$ as shown in Fig. 1.1.

On the other hand, it follows from the boundary conditions of $N_\alpha(\xi)$ on the ∂Q and the trace theorem for boundary layer that

$$\|u^\varepsilon - u^{\varepsilon,b}\|_{1,\Omega_1} \leq C\|u^\varepsilon - u^0\|_{1/2,\partial\Omega_0\cap\partial\Omega_1} \leq C\|u^\varepsilon - u_1^\varepsilon\|_{1,\Omega_0^\varepsilon} \leq C\cdot\varepsilon^{1/2} \tag{5.9}$$

For $s\geq 2$, if $x\in\Omega_0^\varepsilon$, then we have the following equation which holds in the sense of distributions:

$$\begin{aligned}\mathcal{L}_\varepsilon(u^\varepsilon - U_s^\varepsilon) \equiv \mathcal{L}_\varepsilon(u^\varepsilon - u_s^\varepsilon) &= \varepsilon^{s-1}\sum_{\alpha_1,\dots,\alpha_{s+1}=1}^{n} a_{\alpha_1 j}(\xi)\frac{\partial N_{\alpha_2\cdots\alpha_{s+1}}(\xi)}{\partial\xi_j}D^\alpha u^0(x)\\
&+\varepsilon^{s-1}\sum_{\alpha_1,\dots,\alpha_{s+1}=1}^{n}\frac{\partial}{\partial\xi_i}\Big(a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_{s+1}}(\xi)\Big)D^\alpha u^0(x)\\
&+\varepsilon^{s-1}\sum_{\alpha_1,\dots,\alpha_{s+1}=1}^{n} a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_{s+1}}(\xi)D^\alpha u^0(x)\\
&+\varepsilon^{s}\sum_{\alpha_1,\dots,\alpha_{s+2}=1}^{n} a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_{s+2}}(\xi)D^\alpha u^0(x)\\
&= F_1(\varepsilon,\xi,x) + R_c(\varepsilon,\xi,x)\end{aligned} \tag{5.10}$$

If $x\in\Omega_1$, combining (2.8) and (2.9), then it holds

$$\mathcal{L}_\varepsilon(u^\varepsilon - U_s^\varepsilon) = \mathcal{L}_\varepsilon(u^\varepsilon - u^{\varepsilon,b}) = f(x) - f(x) = 0 \tag{5.11}$$

Setting the text function $v(x) = u^\varepsilon - U_s^\varepsilon \in H^1(\Omega^\varepsilon, \partial\Omega)$, and combining (4.10) and (4.11),one yields

$$\begin{aligned}\int\limits_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon})\frac{\partial(u^\varepsilon - U_s^\varepsilon)}{\partial x_i}\frac{\partial(u^\varepsilon - U_s^\varepsilon)}{\partial x_j} &= \int\limits_{\Omega^\varepsilon} F_1 \cdot (u^\varepsilon - U_s^\varepsilon)dx \\ &\quad + \int\limits_{\partial\Omega_0\cap\partial\Omega_1} \Big[\sigma_\varepsilon(U_s^\varepsilon)\Big](u^\varepsilon - U_s^\varepsilon)d\Gamma_x \\ &\quad - \sum_{z\in T_\varepsilon}\sum_{l=1}^{s}\int\limits_{\partial E_z} \sigma_\xi(N_{\alpha_1\cdots\alpha_l})D^\alpha u^0(x) \\ &\qquad \cdot(u^\varepsilon - U_s^\varepsilon)d\Gamma_x \\ &\quad - \sum_{z\in T_\varepsilon}\int\limits_{\partial E_z} R_s \cdot (u^\varepsilon - U_s^\varepsilon)d\Gamma_x\end{aligned}$$

From (4.1) and (4.2), we have

$$\begin{aligned}\|u^\varepsilon - U_s^\varepsilon\|_{1,\Omega^\varepsilon} &\le C\Big[\varepsilon^{min(3,s-1)} + \delta^{\beta_0/2} + \Big(\int\limits_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(U_s^\varepsilon)]\Big)^2 dx\Big)^{1/2}\Big] \\ &\le C\Big[\varepsilon^{min(3,s-1)} + \Big(\int\limits_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(U_s^\varepsilon)]\Big)^2 dx\Big)^{1/2}\Big] \qquad (5.12)\end{aligned}$$

Besides, following along the lines of proofs of (5.8) and (5.9)(see, [7,18]), we have

$$\|u^\varepsilon - U_s^\varepsilon\|_{1,\Omega^\varepsilon} \le C \cdot \varepsilon^{1/2}, \quad s \ge 2 \qquad (5.13)$$

Combining (5.12) and (5.13) leads

$$\|u^\varepsilon - U_s^\varepsilon\|_{1,\Omega^\varepsilon} \le C \cdot min(\varepsilon^{1/2}, \zeta_s), \quad s \ge 2 \qquad (5.14)$$

where $\zeta_s = \varepsilon^{min(3,s-1)} + \Big(\int\limits_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(U_s^\varepsilon)]\Big)^2 dx\Big)^{1/2}$.

Therefore the proof of Theorem 5.2 is complete.

6 The Regularity Estimate and Finite Element Computation of Boundary Layer

We first give a priori estimate result for the solution of boundary layer (2.8).

Theorem 6.1 *Let $u^{\varepsilon,b}(x)$ be the weak solution of problem (2.8). If $a_{ij} \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $\bar{u} \in H^{1/2}(\partial\Omega)$, then it holds*

$$\|u^{\varepsilon,b}\|_{1,\Omega_1} \le C\Big(\|f\|_{0,\Omega} + \|\bar{u}\|_{1/2,\partial\Omega}\Big) \qquad (6.1)$$

where C is independent of ε, $u^{\varepsilon,b}$, f, $\bar{u}$.

Proof It is easy to verify that $(u^{\varepsilon,b}-u^0) \in H^1(\Omega_1, \Gamma^* \cup \partial\Omega)$ satisfies the following integrating identity:

$$\int\limits_{\Omega_1} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon,b}(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}dx = \int\limits_{\Omega_1} f(x)v(x)dx, \quad \forall v \in H^1(\Omega_1, \Gamma^* \cup \partial\Omega) \tag{6.2}$$

where $H^1(\Omega_1, \Gamma^* \cup \partial\Omega) = \{v \in H^1(\Omega_1), v|_{\Gamma^* \cup \partial\Omega} = 0\}$, and $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$ shown in (2.8).

It follows from the conditions (A_2) and (A_3), and the trace theorem that

$$\begin{aligned}\sigma\|u^{\varepsilon,b}-u^0\|^2_{1,\Omega_1} &\le \int\limits_{\Omega_1} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial(u^{\varepsilon,b}-u^0)}{\partial x_j}\frac{\partial(u^{\varepsilon,b}-u^0)}{\partial x_i} \\ &= \int\limits_{\Omega_1} f(x)(u^{\varepsilon,b}(x)-u^0(x))dx \\ &\quad -\int\limits_{\Omega_1} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^0}{\partial x_j}\frac{\partial(u^{\varepsilon,b}-u^0)}{\partial x_i} \\ &\le C\Big(\|f\|_{0,\Omega_1}\|u^{\varepsilon,b}-u^0\|_{0,\Omega_1} \\ &\quad +\|u^0\|_{1,\Omega_1}\|u^{\varepsilon,b}-u^0\|_{1,\Omega_1}\Big) \\ &\le C\Big(\|f\|_{0,\Omega_1}+\|u^0\|_{1,\Omega_1}\Big)\|u^{\varepsilon,b}-u^0\|_{1,\Omega_1}\end{aligned}$$

Therefore

$$\|u^{\varepsilon,b}-u^0\|_{1,\Omega_1} \le C\Big(\|f\|_{0,\Omega_1}+\|u^0\|_{1,\Omega_1}\Big)$$

and

$$\|u^{\varepsilon,b}\|_{1,\Omega_1} \le C\Big(\|f\|_{0,\Omega_1}+\|u^0\|_{1,\Omega_1}\Big) \le C\Big(\|f\|_{0,\Omega}+\|\bar{u}\|_{1/2,\partial\Omega}\Big)$$

We now consider the boundary value problems only in the two dimensional case without loss of generality.

Theorem 6.2 *Suppose* Ω_0, $\Omega_1 = \Omega\setminus\overline{\Omega}_0 \subset R^2$ *are shown in Figs. 2.1 and 2.2. Let* $u^{\varepsilon,b}(x)$ *be the weak solution of boundary value problem (2.8) with pure Dirichlet boundary conditions. If* $a_{ij}(\frac{x}{\varepsilon}) \in C^0(\overline{\Omega}^\varepsilon)$, $\nabla_\xi a_{ij} \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $\bar{u} \in W^{3/2,p}(\partial\Omega)$, *then there exists* $1 < p_0 < +\infty$, *such that*

$$u^{\varepsilon,b} \in W^{2,p}(\Omega_1), \quad 1 < p \le p_0 \tag{6.3}$$

$$\|u^{\varepsilon,b}\|_{2,p,\Omega_1} \le C_1(p,\varepsilon)\cdot(\|f\|_{0,p,\Omega}+\|\bar{u}\|_{2,p,\Omega}) \tag{6.4}$$

where $C_1(p,\varepsilon) \le C\varepsilon^{-2}$, *and* C *is a constant independent of* ε.

Before giving the proof of Theorem 6.2, we shall consider the boundary value problems over a concave domain $\Omega' \subset R^2$:

$$\begin{cases} -\triangle u = f(x) & \text{in} \quad \Omega' \\ u(x) = 0 & \qquad \partial\Omega' \end{cases} \tag{6.5a}$$

Denoted by $\{\sigma_j\}_{j=1}^M$ the angular points of Ω', respectively, and where $\beta_j\pi, \quad j = 1, \ldots M$ are the corresponding internal angles, i.e.

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_M, \quad \gamma_j = \frac{1}{\beta_j}$$

Given $1 < \beta_M \leq 2$ implies that $\frac{1}{2} \leq \gamma_M < 1$. Let

$$V_j = \{x \in \Omega' : |x - \sigma_j| < r_j\}, \quad j = 1, \ldots, M \tag{6.5b}$$

satisfy

$$V_i \cap V_j = \emptyset, \quad V_0 = \Omega' \setminus \cup_{j=1}^M \overline{V}_j \tag{6.5c}$$

Lemma 6.1 *[12] Suppose u is the unique solution of problem (6.5 a), and* $f \in L^2(\Omega)$, *then it holds*

$$u(x) = \sum_{j=1}^{M} c_j(f)u_j(x) + U(x) \tag{6.6}$$

where $U \in H^2(\Omega') \cap H_0^1(\Omega'), \quad \|U\|_2 \leq C\|f\|_0$, *and the constants* $c_j(f)$ *satisfy* $|c_j(f)| \leq C\|f\|_0$, *and* $u_j(x)$, $j = 1, 2. \ldots, M$ *are some functions independent of* $f(x)$, $u(x)$, *and satisfy the following conditions*

(H_1) If $\gamma_j > 1$, *then* $u_j(x) \equiv 0$.
(H_2) If $x \notin V_j$, $u_j(x) \equiv 0$.
(H_3) If $\frac{1}{2} < \gamma_j < 1$, *then there exists the following formula in a neighborhood of* σ_j*:*

$$u_j = \rho^{\gamma_j} sin\gamma_j\theta, \quad if \quad (\rho, \theta) \in V_j \tag{6.7}$$

where $V_j = \{x \in \Omega' : \quad |x - \sigma_j| < r_j\}, \quad j = 1, 2, \ldots, M.$

(6.7) implies that $|D^k u| \leq C\rho^{\gamma_j - |k|}$ *in a neighborhood of* σ_j*.*

Let us turn to proof of Theorem 6.2.

According to the finite covering theorem, there exist a finite number of points $P_1, \ldots P_t$, and the corresponding neighborhoods $\mathcal{O}_l, \quad l = 1, \ldots t$, such that

(i) $\cup_{l=1}^t \mathcal{O}_l \supset \overline{\Omega}_1$;

(ii) $diam(\mathcal{O}_l) \leq \varepsilon R_0, \quad R_0$ will be chosen below.

(iii) $\mathcal{I}_i = \{j : \mathcal{O}_j \cap \mathcal{O}_i \neq \emptyset, \}, \quad \sigma(\mathcal{I}_i) \leq s_0$, where $\sigma(\mathcal{I}_i)$ denote the numbers of elements in $\mathcal{I}_i$, respectively, $i = 1, \ldots t$ and s_0 is a constant.

It follows from the partition of unity theorem that there exist $\phi_l(x) \in C_0^\infty(R^n)$, $l = 1, \ldots t$, such that $0 \leq \phi_l(x) \leq 1$, supp$\phi_l \subset \mathcal{O}_l$, and $\sum_{l=1}^t \phi_l(x) \equiv 1, \quad$ in $\quad \Omega_1$

If let $u^{\varepsilon,b} = \sum_{l=1}^{t} u_l^{\varepsilon,b}, \quad u_l^{\varepsilon,b} = \phi_l \cdot u^{\varepsilon,b}$, then we have

$$\mathcal{L}_\varepsilon u_l^{\varepsilon,b} = \phi_l \cdot \mathcal{L}_\varepsilon u^{\varepsilon,b} + \eta_l \tag{6.8}$$

where

$$\begin{aligned} \eta_l = & -\frac{\partial \phi_l}{\partial x_j}[\frac{\partial}{\partial x_i}\Big(a_{ij}\left(\frac{x}{\varepsilon}\right)\Big)u^{\varepsilon,b} + a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon,b}}{\partial x_j}] \\ & -a_{ij}\left(\frac{x}{\varepsilon}\right)[\frac{\partial \phi_l}{\partial x_i}\frac{\partial u^{\varepsilon,b}}{\partial x_j} + u^{\varepsilon,b}\frac{\partial^2 \phi_l}{\partial x_i \partial x_j}] \end{aligned} \tag{6.9}$$

and $\|\phi_l\|_{2,p} \leq C\varepsilon^{-2}$, consequently

$$\|\eta_l\|_{0,p,\mathcal{O}_l\cap\Omega_1} \leq C\frac{1}{\varepsilon^2}\|u^{\varepsilon,b}\|_{1,p,\mathcal{O}_l\cap\Omega_1} \tag{6.10}$$

$\forall R > 0$, let $\omega_\varepsilon(R) = \max\limits_{i,j} \max\limits_{|x-x'|<\varepsilon R} |a_{ij}(\frac{x}{\varepsilon}) - a_{ij}(\frac{x'}{\varepsilon})|, \quad x, x' \in \Omega_1$

For any fixed $x_0 \in \mathcal{O}_l \cap \Omega_1$, set $A^\varepsilon = (a_{ij}(\frac{x_0}{\varepsilon}))$, it follows from (A_3) that there exists a orthogonal matrix T such that

$$TA^\varepsilon T' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$$

where T' denotes the transpose of a matrix T.

From (A_2), we have $\lambda_i \geq \sigma > 0, \quad i = 1, 2$. If let $B = D^{-1/2}T$, then $BA^\varepsilon B' = I$, and $\|B\|_2 = \|D^{-1/2}\|_2 \leq \sigma^{-1/2}$. Hence

$$\|B^{-1}\|_2^2 = \|D\|_2 \leq \sum_{i=1}^{2} \lambda_i = \sum_i a_{ii}(\frac{x_0}{\varepsilon}) \leq M_0$$

where M_0 is a constant independent of ε.

If let $\hat{\mathcal{O}}_l = B(\mathcal{O}_l \cap \Omega_1)$, then $\hat{v}(y) = v(B^{-1}y) \in W^{2,p}(\hat{\mathcal{O}}_l)$, for any $v \in W^{2,p}(\mathcal{O}_l \cap \Omega_1)$, where p will be determined below.

$$C\|v\|_{2,p,\mathcal{O}_l\cap\Omega_1} \leq \|\hat{v}\|_{2,p,\hat{\mathcal{O}}_l} \leq C'\|v\|_{2,p,\mathcal{O}_l\cap\Omega_1} \tag{6.11}$$

Let

$$\begin{aligned} g(x) = -a_{ij}\left(\frac{x_0}{\varepsilon}\right)\frac{\partial^2 u_l^{\varepsilon,b}}{\partial x_i \partial x_j} &= -(a_{ij}\left(\frac{x_0}{\varepsilon}\right) - a_{ij}\left(\frac{x}{\varepsilon}\right))\frac{\partial^2 u_l^{\varepsilon,b}}{\partial x_i \partial x_j} - a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_l^{\varepsilon,b}}{\partial x_i \partial x_j} \\ = -(a_{ij}\left(\frac{x_0}{\varepsilon}\right) - a_{ij}\left(\frac{x}{\varepsilon}\right))&\frac{\partial^2 u_l^{\varepsilon,b}}{\partial x_i \partial x_j} + \phi_l(x)f(x) + \eta_l(x) + \frac{\partial}{\partial x_i}(a_{ij}\left(\frac{x}{\varepsilon}\right))\frac{\partial u_l^{\varepsilon,b}}{\partial x_j} \end{aligned}$$

From (6.10), (6.11) and condition $\nabla_\xi a_{ij} \in L^\infty(\Omega)$, one derives

$$\begin{aligned} \|g\|_{0,p,\mathcal{O}_l\cap\Omega_1} \leq & \ \omega_\varepsilon(R)\|u_l^{\varepsilon,b}\|_{2,p,\mathcal{O}_l\cap\Omega_1} \\ & + C\frac{1}{\varepsilon^2}\|u^{\varepsilon,b}\|_{1,p,\mathcal{O}_l\cap\Omega_1} + C\|f\|_{0,p,\mathcal{O}_l\cap\Omega_1} \end{aligned} \tag{6.12}$$

On the other hand, if let $\hat{u}_l^{\varepsilon,b}(y) = u_l^{\varepsilon,b}(B^{-1}y), \quad \hat{g}(y) = g(B^{-1}y)$, then

$$a_{ij}\left(\frac{x_0}{\varepsilon}\right)\frac{\partial^2 u_l^{\varepsilon,b}}{\partial x_i \partial x_j} = \Delta\hat{u}_l^{\varepsilon,b}(y), \quad y = Bx$$

and consequently

$$\Delta\hat{u}_l^{\varepsilon,b}(y) = \hat{g}(y)$$

It follows from Lemma 6.1 and (6.7) that

$$\begin{aligned} \|\hat{u}_l^{\varepsilon,b}\|_{2,p,\hat{\mathcal{O}}_l} &\le C(p)\{\|\Delta\hat{u}_l^{\varepsilon,b}\|_{0,p,\hat{\mathcal{O}}_l} + \|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\} \\ &= C(p)\{\|\hat{g}\|_{0,p,\hat{\mathcal{O}}_l} + \|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\} \end{aligned} \tag{6.13}$$

where $1 < p \le p_0 = \dfrac{2\beta'_M}{2\beta'_M - 1} < +\infty$, and β'_M depends on not only the maximum internal angle of Ω_1, but the transformation B also.

Combining (6.11), (6.12) and (6.13), it leads

$$\begin{aligned} \|u_l^{\varepsilon,b}\|_{2,p,\mathcal{O}_l\cap\Omega_1} \le C(p)\{&\omega_\varepsilon(R)\|u_l^{\varepsilon,b}\|_{2,p,\mathcal{O}_l\cap\Omega_1} + \frac{1}{\varepsilon^2}\|u^{\varepsilon,b}\|_{1,p,\mathcal{O}_l\cap\Omega_1} \\ &+\|f\|_{0,p,\mathcal{O}_l\cap\Omega_1} + \|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\} \end{aligned}$$

If $a_{ij}(\frac{x}{\varepsilon}) \in C^0(\overline{\Omega}_1)$, then there exists a constant $R_0 > 0$ such that

$$\omega_\varepsilon(R) < \frac{1}{3C(p)} \quad \text{for} \quad 0 < R < R_0$$

Hence

$$\begin{aligned} \|u_l^{\varepsilon,b}\|_{2,p,\mathcal{O}_l\cap\Omega_1} &\le C(p)\varepsilon^{-2}\{\|u_l^{\varepsilon,b}\|_{1,p,\mathcal{O}_l\cap\Omega_1} + \|f\|_{0,p,\mathcal{O}_l\cap\Omega_1} + \|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\} \\ &\le C(p)\varepsilon^{-2}\{\|f\|_{0,p,\mathcal{O}_l\cap\Omega_1} + \|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\} \end{aligned}$$

Therefore

$$\begin{aligned} \|u^{\varepsilon,b}\|_{2,p,\Omega_1} = |\sum_{l=1}^{t} u_l^{\varepsilon,b}\|_{2,p,\Omega_1} &\le \sum_{l=1}^{t}\|u_l^{\varepsilon,b}\|_{2,p,\mathcal{O}_l\cap\Omega_1} \\ &\le C(p)\varepsilon^{-2}\{\|f\|_{0,p,\Omega} + \|u^0\|_{2,p,\Omega}\} \\ &\le C_1(p,\varepsilon)\{\|f\|_{0,p,\Omega} + \|\bar{u}\|_{2,p,\Omega}\} \end{aligned}$$

In finite element computation of boundary layer, one needs to solve the modified equation as follows:

$$\begin{cases} \mathcal{L}_\varepsilon\tilde{u}^{\varepsilon,b} \equiv -\dfrac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial\tilde{u}^{\varepsilon,b}(x)}{\partial x_j}) = f(x), \quad x \in \Omega_1 \\ \tilde{u}^{\varepsilon,b}(x) = \bar{u}(x), \quad x \in \partial\Omega \\ \tilde{u}^{\varepsilon,b}(x) = \tilde{u}_h^0(x) \quad x \in \Gamma^* = \partial\Omega_0 \cap \partial\Omega_1 \end{cases} \tag{6.14}$$

where $\tilde{u}_h^0(x)$ is the finite element approximate solution of $\tilde{u}^0(x)$, and h is the size of mesh for domain Ω.

Let $\mathcal{F}^{h_1} = \{e\}$ be a regular family of triangulations for subdomain $\Omega_1 = \Omega\backslash\overline{\Omega}_0$ as shown in Fig. 2.2, $h_1 = \max\limits_{e\in\mathcal{F}^{h_1}}\{h_e\}, \quad 0 < \dfrac{h_1}{\varepsilon^2} << 1.$

Define a linear finite element space

$$V^{\varepsilon}_{h_1}(\Omega_1) = \{v \in C^0(\overline{\Omega}_1) : v|_e \in P_1(e), v|_{\partial\Omega_0\cap\partial\Omega_1} = \tilde{u}^0_h(x), v|_{\partial\Omega} = \bar{u}(x)\} \quad (6.15)$$

Theorem 6.3 *Let $u^{\varepsilon,b}(x), \tilde{u}^{\varepsilon,b}(x)$ be the weak solutions of problems (2.8) and (6.14) with pure Dirichlet boundary conditions, respectively, and let $\tilde{u}^{\varepsilon,b}_{h_1}(x)$ be the finite element solution of $\tilde{u}^{\varepsilon,b}(x)$ in $V^{\varepsilon}_{h_1}(\Omega_1)$. If $f \in L^p(\Omega)$, $\bar{u} \in W^{3/2,p}(\partial\Omega)$, then it holds*

$$\|u^{\varepsilon,b} - \tilde{u}^{\varepsilon,b}_{h_1}\|_{1,p,\Omega_1} \leq C\Big\{h_0 + h + (\frac{h_1}{\varepsilon^2})\Big\}\Big(\|f\|_{0,p,\Omega} + \|\bar{u}\|_{3/2,p,\partial\Omega}\Big) \quad (6.16)$$

where $1 < p \leq p_0 < +\infty$, and h_1 is the mesh size of Ω_1.

Proof It follows from Theorem 6.2 that there exists a constant $1 < p_0 < +\infty$ such that, if $1 < p \leq p_0$, then $\tilde{u}^{\varepsilon,b} \in W^{2,p}(\Omega_1)$, and

$$\begin{aligned}\|\tilde{u}^{\varepsilon,b} - \tilde{u}^{\varepsilon,b}_{h_1}\|_{1,p,\Omega_1} &\leq Ch_1\|\tilde{u}^{\varepsilon,b}\|_{2,p,\Omega_1}\\ &\leq C(\frac{h_1}{\varepsilon^2})\Big\{\|f\|_{0,p,\Omega} + \|\bar{u}\|_{3/2,p,\partial\Omega} + \|\tilde{u}^0\|_{2,p,\Omega}\Big\}\end{aligned}$$

Using the triangular inequality, one derives

$$\begin{aligned}\|u^{\varepsilon,b} - \tilde{u}^{\varepsilon,b}_{h_1}\|_{1,p,\Omega_1} &\leq \|u^{\varepsilon,b} - \tilde{u}^{\varepsilon,b}\|_{1,p,\Omega_1} + \|\tilde{u}^{\varepsilon,b} - \tilde{u}^{\varepsilon,b}_{h_1}\|_{1,p,\Omega_1}\\ &\leq \|u^0 - \tilde{u}^0_h\|_{1,p,\Omega}\\ &\quad +C(\frac{h_1}{\varepsilon^2})\{\|f\|_{0,p,\Omega} + \|\bar{u}\|_{3/2,p,\partial\Omega} + \|\tilde{u}^0\|_{2,p,\Omega}\}\\ &\leq C\{\|u^0 - \tilde{u}^0\|_{1,p,\Omega} + \|\tilde{u}^0 - \tilde{u}^0_h\|_{1,p,\Omega}\}\\ &\quad +C(\frac{h_1}{\varepsilon^2})\{\|f\|_{0,p,\Omega} + \|\bar{u}\|_{3/2,p,\partial\Omega} + \|\tilde{u}^0\|_{2,p,\Omega}\}\\ &\leq C\Big\{h_0 + h + (\frac{h_1}{\varepsilon^2})\Big\}\Big(\|f\|_{0,p,\Omega} + \|\bar{u}\|_{3/2,p,\partial\Omega}\Big)\end{aligned}$$

7 Multiscale Finite Element Method and Numerical Results

To summarize the theoretical results in the previous sections, one can conclude that the multiscale finite element method consists of the following steps:

Step 1: Compute successively the functions $N_{\alpha_1}(\xi), \ldots, N_{\alpha_1\cdots\alpha_s}(\xi), \quad s \geq 1$ given in (2.2),(2.3) and (2.5) in the unit cell $Q \cap \omega$.

Step 2: Solve numerically the homogenized equation (2.6) in the whole domain Ω in a coarse mesh.

Step 3: Solve numerically the boundary layer (2.8) in a domain Ω_1 in a refined mesh.

Step 4: Calculate successively the higher-order partial derivatives of the solution $u^0(x)$ of the homogenized equation (2.6) by using higher-order difference quotients.

To do so, we need to implement the partitions for $Q\cap\omega$, Ω, and Ω_1, respectively. Denoted by h_0, h, h_1 the sizes of the corresponding meshes.

It should be emphasized that we cannot directly take higher-order partial derivatives for the finite element solution $u_h^0(x)$ of the solution $u^0(x)$ for the homogenized equation (2.6). Otherwise, one will get some meaningless results, for example, $D^\alpha u_h^0(x) = 0$, if $|\alpha| \geq 2$ for the linear finite element. We now present the finite difference method for calculating the higher-order partial derivatives of the solution $u^0(x)$. First, define the first-order difference quotient as follows:

$$\delta_{x_i} u_h^0(N_p) = \frac{1}{\tau(N_p)} \sum_{e\in\sigma(N_p)} [\frac{\partial u_h^0}{\partial x_i}]_e(N_p) \tag{7.1}$$

where $\sigma(N_p)$ is the set of elements with node N_p, $\tau(N_p)$ is the number of elements of $\sigma(N_p)$, $u_h^0(x)$ is the finite element solution of $u^0(x)$ in $S^h(\Omega)$, $[\frac{\partial u_h^0}{\partial x_i}]_e(N_p)$ is the value of the derivative $\frac{\partial u_h^0}{\partial x_i}$ at node N_p associated with element e.

Secondly, the higher-order difference quotients can be given

$$\delta^l_{x_{\alpha_1},\dots x_{\alpha_l}} u_h^0(N_p) = \frac{1}{\tau(N_p)} \sum_{e\in\sigma(N_p)} \left[\sum_{j=1}^{d} \delta^{l-1}_{x_{\alpha_1}\dots x_{\alpha_{l-1}}} u_h^0(P_j) \frac{\partial \psi_j}{\partial x_{\alpha_l}}\right]_e (N_p) \tag{7.2}$$

where d is the number of nodes on e, P_j are the nodes of e, $\psi_j(x)$ are Lagrange's type shape functions, $j = 1, 2\dots d$.

In a word, a multiscale finite element method can be written as

$$U^{\varepsilon,h_0,h}_{s,h_1}(N_p) = \begin{cases} u_h^0(N_p) + \sum\limits_{l=1}^{s} \varepsilon^l \sum\limits_{\alpha_1,\dots,\alpha_l=1}^{n} N^{h_0}_{\alpha_1\cdots\alpha_l}(\xi(N_p))\delta^l_{x_{\alpha_1}\dots x_{\alpha_l}} u_h^0(N_p), & N_p \in \overline{\Omega}_0^\varepsilon \\ u^{\varepsilon,b}_{h_1}(N_p), \quad N_p \in \Omega_1, \quad s \geq 1 & \end{cases} \tag{7.3}$$

Next let us show some numerical results.

Example 7.1 Consider the mixed boundary value problem of second order elliptic equation with highly oscillatory coefficients as follows:

$$\begin{cases} \mathcal{L}_\varepsilon u^\varepsilon \equiv -\frac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon(x)}{\partial x_j}) = f(x), \quad x \in \Omega^\varepsilon \\ \sigma_\varepsilon(u^\varepsilon) \equiv -\nu_i a_{ij}(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon}{\partial x_j} = 0, \quad x \in S_\varepsilon \\ u^\varepsilon(x) = 0 \quad x \in \partial\Omega \end{cases} \tag{7.4}$$

where Ω^{ε} is a bounded perforated domain as shown in Fig. 7.1, S_{ε} is the surface of holes, and $\partial\Omega$ is the outer boundary of domain Ω^{ε}. Notice that Ω^{ε} is not the union of entire periodic cells, generally speaking, the truncated function $u_s^{\varepsilon}(x)$ does not satisfy the boundary conditions for the solution $u^{\varepsilon}(x)$ on the boundary $\partial\Omega$, so we need to construct the boundary layer. The unit cell $Q\cap\omega$ is shown in Fig. 7.2, $\varepsilon = \frac{1}{12}$.

Case 1: $a_{ij0} = 1.0\delta_{ij}, \quad a_{ij1} = 20.0\delta_{ij}, \quad f(x) = 10;$
Case 2: $a_{ij0} = 1.0\delta_{ij}, \quad a_{ij1} = 114.0\delta_{ij}, \quad f(x) = 2e^{x^2+y^2} + log(x+y+10);$
Case 3: $a_{ij0} = 1.0\delta_{ij}, \quad a_{ij1} = 0.008\delta_{ij}, \quad f(x) = 10.$

where δ_{ij} is a Kronecker delta.

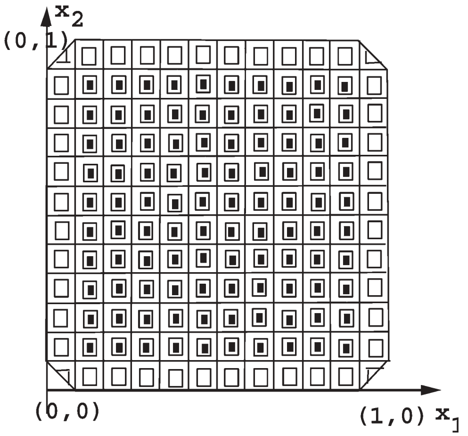


Fig. 7.1 The whole domain Ω^{ε}

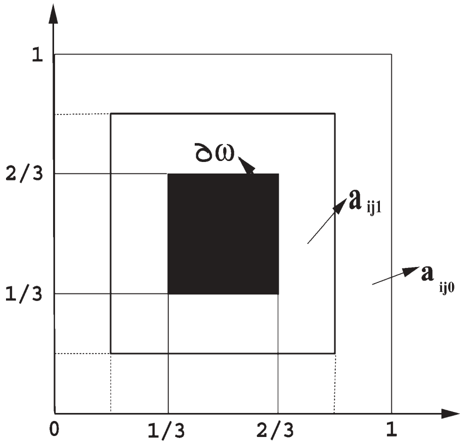


Fig. 7.2 The unit cell $Q \cap \omega$

Table 1 The comparison of computational amount

	original equation	cell problem	homogenized equation	boundary layer
elements	37696	1296	10224	3024
nodes	19813	1369	5245	1764

Table 2 The comparison of computational results

	$\frac{\|e_0\|_{L^2}}{\|u^0\|_{L^2}}$	$\frac{\|e_1\|_{L^2}}{\|U_1^\varepsilon\|_{L^2}}$	$\frac{\|e_2\|_{L^2}}{\|U_2^\varepsilon\|_{L^2}}$	$\frac{\|e_0\|_{H^1}}{\|u^0\|_{H^1}}$	$\frac{\|e_1\|_{H^1}}{\|U_1^\varepsilon\|_{H^1}}$	$\frac{\|e_2\|_{H^1}}{\|U_2^\varepsilon\|_{H^1}}$
Case 1	0.046468	0.009933	0.010026	0.191074	0.056482	0.055872
Case 2	0.083461	0.055794	0.055989	0.250699	0.086833	0.085923
Case 3	0.076103	0.056306	0.052530	0.395522	0.335061	0.304321

We first state that, in order to compute numerically the original problem in a refined mesh, and to compare with the related computational results, we only do numerical experiment for a simple model problems (7.4). If we do not need to compute numerically the original problem in a very refined mesh, then we can treat much more complex problems, e.g. heat transfer problems in 3-D porous media. We now implement the triangular partition for Ω^ε, which is such that the discontinuities of the coefficients a_{ij} coincide with sides of the triangles. The number of triangles is 37696.

In order to solve numerically the cell problems (2.2),(2.3)and (2.5), the homogenized equation (2.6) and boundary layer (2.8), we implement the rectangular partition for $Q \cap \omega$, the triangular partition for Ω, and the triangular partition for Ω_1, respectively. The sizes of the corresponding meshes are respectively $h_0 = \frac{1}{36}$, $h = \frac{1}{48}$, $h_1 = \frac{1}{48}$.

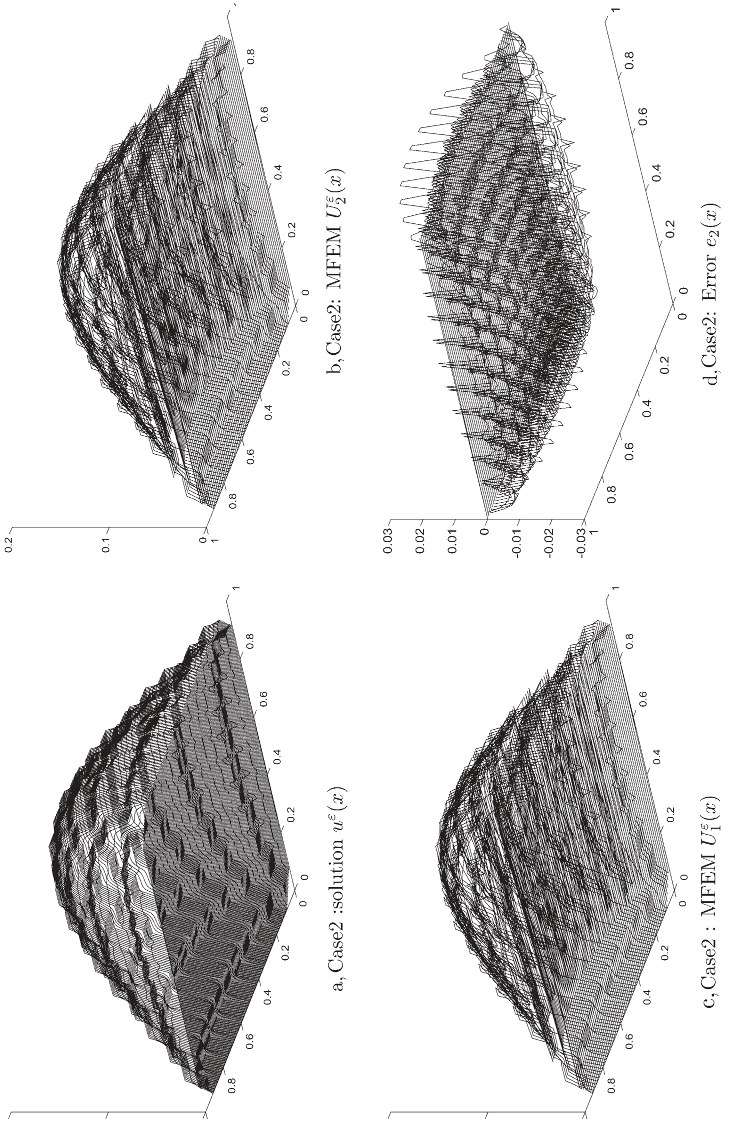
For simplicity, let $u^\varepsilon(x)$ denote the finite element solution for the solution of problem (7.4) in a refined mesh, and $u^0(x)$ denote the finite element solution for the corresponding homogenized equation (2.6) in a coarse mesh. $U_1^\varepsilon(x)$, $U_2^\varepsilon(x)$ are respectively the first-order and the second-order multiscale finite element solutions calculated by the formulas (7.3). Set $e_0 = u^\varepsilon - u^0$, $e_1 = u^\varepsilon - U_1^\varepsilon$, $e_2 = u^\varepsilon - U_2^\varepsilon$.

Some numerical results are given in **Table 2**.

Figs. 7.3a-7.3d show some numerical results in **Case 2**.

Remark 7.1 Observe some numerical results shown in **Table 2**, we can deduce that, if $a_{ij}^\varepsilon(\frac{x}{\varepsilon})$ in different parts of the unit cell $Q \cap \omega$ are close, then homogenization has the better accuracy (see, **Case 1**), but if the differences are large, then homogenization is not good(see, **Case 2,3**). In the latter cases, one needs to use the first-order, or even the higher-order asymptotic methods presented in this paper.

Remark 7.2 Finally, it should be emphasized that the proposed method in this paper is suitable for the subdivided periodic domains(or also called local periodic structures). The reason is that, generally speaking, the asymptotic solutions $u_s^\varepsilon \notin H^1(\Omega^\varepsilon)$ if they are obtained by some classical methods(cf. [4,18]). However, the proposed method in this paper can guarantee the asymptotic solutions $u_s^\varepsilon \in H^1(\Omega^\varepsilon)$, due to the homogeneous Dirichlet boundary conditions on ∂Q.



a, Case2 :solution $u^{\varepsilon}(x)$

b, Case2: MFEM $U_2^{\varepsilon}(x)$

c, Case2 : MFEM $U_1^{\varepsilon}(x)$

d, Case2: Error $e_2(x)$

Fig. 7.3

Acknowledgements Author of this paper thanks to The State Key Laboratory of Science-Engineering Computing (P.R. China). For various improvement, such as style and presentations, I am especially indebted to referees.

References

1. Acerbi, E., Chiado-Piat, V., Dal Maso, G., Pecivale, D.: An extension theorem for connected sets and homogenization in general periodic domains. Nonlinear Analysis, Theory Meth. Appl. **18**, 418–496 (1992)
2. Allaire, G., Murat, F.: Homogenization of the Neumann problem with nonisolated holes. Asymptotic Anal. **7**, 81–95 (1993)
3. Antonić, N.: et al. (eds.) Multiscale Problems in Science and Technology, Challenges to Mathematical Analysis and Perspectives, Springer-Verlag, Berlin, 2002
4. Bensoussan, A., Lions, J.L., Papanicolaou, G.: Asymptotic Analysis of Periodic Structures, North-Holland, Amsterdam,1978
5. Cao, L.Q., Cui, J.Z.: Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problem for second order elliptic equations in perforated domains, Numer. Math. **96**, 525–581 (2004)
6. Cao, L.Q.: Asymptotic expansion and convergence theorem of control and observation on the boundary for second-order elliptic equation with highly oscillatory coefficients, Math. Models & Meth. Appl. Sci. **14**(3), 417–437 (2004)
7. Cioranescu, D., Jean Paulin, J.S.: Homogenization of Reticulated Structures, Springer-Verlag, New York, 1998
8. Conca, C., Natesan, S.: Numerical methods for elliptic partial differential equations with rapidly oscillating coefficients. Comput. Methods. Appl. Mech. Engrg. **192**, 47–76 (2003)
9. Conca, C., Natesan, S., Vanninathan, M.: Numerical solution of elliptic partial differential equations by Bloch waves method, VII CEDYA, Congress on Differential Equations and Applications/VII CMA: Congress on Applied Mathematics, Salamanca, Ferragut et al. eds, 2001 pp. 63–83
10. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order (2nd ed), Springer-Verlag, Berlin and New York, 1983
11. Gilbarg, D., Hörmander, L.: Intermediate Schauder Estimates. Archiev for Rational Mechanics and Analysis **74,** 297–318 (1980)
12. Grisvard, P.: Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain, Numerical Solution of Partial Differential Equations-III (B. Hubbard Editor) Academic Press, New York, 1976 pp. 207–274
13. Hornung, U.: Homogenization and Porous Media, Springer-Verlag, New York, 1997
14. Hou, T.Y., Wu, X.H., Cai, Z.: Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients. Math. Comp. **68**, 913–943 (1999)
15. Jikov, V.V., Kozlov, S.M., Oleinik, O.A.: Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994
16. Lions, J.L.: Some methods for mathematical analysis of systems and their controls, Science Press, Beijing, 1981
17. Lions, J.L.: Asymptotic expansions in perforated media with a periodic structure, the rocky mountain. J. Math. **10**(1), 125–144 (1980)
18. Oleinik, O.A., Shamaev, A.S., Yosifian, G.A.: Mathematical Problems in Elasticity and Homogenization, North-Holland, Amsterdam, 1992
19. Walker, J.S.: Fourier Analysis, Oxford University Press, New York, Oxford, 1988