

Integrable semi-discretizations and full-discretization of the two dimensional Leznov lattice

Xing-Biao Hu¹ and Guo-Fu Yu^{2*}

¹ LSEC, Institute of Computational Mathematics and Scientific Engineering Computing, Academy of Mathematics and Systems Science,

Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, P.R. CHINA

²Department of Mathematics, Shanghai Jiao Tong University, Shanghai, 200240, P.R. CHINA

Abstract

In this paper, semi-discretizations and full-discretization of the Leznov lattice are investigated via Hirota's bilinear formalism. As a result, two integrable semi-discrete versions and one fully discrete version for the Leznov lattice are found. Bäcklund transformations, nonlinear superposition formulae and Lax pairs for these discrete versions are presented.

Mathematics Subject Classification (2000). 35Q58, 37K40

Keywords. Integrable system, Integrable discretization, Bäcklund transformation, Lax pair

1 Introduction

The nonlinear two-dimensional Leznov lattice given by [1]

$$\frac{\partial^2}{\partial x \partial y} \ln \theta(n) = \theta(n+1)p(n+1) - 2\theta(n)p(n) + \theta(n-1)p(n-1), \quad (1)$$

$$\frac{\partial p(n)}{\partial y} = \theta(n+1) - \theta(n-1), \quad (2)$$

is a special case of the so-called UToda(m_1, m_2) system with $m_1 = 1, m_2 = 2$. If we set the variable transformations $a(n) = p(n+1), c(n) = \theta(n+1)$, then (1) and (2) can be transformed into the form

$$a_y(n) = c(n+1) - c(n-1), \quad (3)$$

$$b_y(n) = a(n-1)c(n-1) - a(n)c(n), \quad (4)$$

$$c_x(n) = c(n)[b(n) - b(n+1)], \quad (5)$$

which is a two-dimensional generalization of the Blaszak-Marciniak lattice[2, 3]

$$a_t(n) = c(n+1) - c(n-1), \quad (6)$$

$$b_t(n) = a(n-1)c(n-1) - a(n)c(n), \quad (7)$$

$$c_t(n) = c(n)[b(n) - b(n+1)]. \quad (8)$$

In 1985, Kupershmidt proposed the following integrable lattice [4]:

$$q_{i,t}(n) = q_i(n)[q_0(n+i) - q_0(n)] + q_{i+1}(n) - q_{i+1}(n-1), \quad 0 \leq i \leq N, \quad (9)$$

$$q_{N,t}(n) = q_N(n)[q_0(n+N) - q_0(n)]. \quad (10)$$

Obviously, when $N = 1$, by the variable transformation $q_1(n) = e^{u(n)-u(n-1)}$, system (9),(10) is nothing but the Toda lattice

$$u_{tt}(n) = e^{u(n+1)-u(n)} - e^{u(n)-u(n-1)}. \quad (11)$$

*Corresponding author

When $N = 2$, by the Miura-like transformation

$$q_0(n) = -b(n-1), \quad q_1(n) = a(n-1)c(n-1), \quad q_2(n) = c(n)c(n-1). \quad (12)$$

the system (9)-(10) can be transformed into the three-field Blaszk-Marciniak lattice (6)-(8). From above observation we can see that two dimensional Toda lattice

$$u_{ts}(n) = e^{u(n+1)-u(n)} - e^{u(n)-u(n-1)} \quad (13)$$

is different from the two-dimensional Leznov lattice since their one-dimensional versions correspond to different N values in the Kupershmidt lattice (9)-(10). For the two-dimensional Toda lattice, its integrable discretization form is the famous fully discrete KP equation (or Hirota-Miwa equation). It is quite natural and reasonable for us to consider integrable discrete versions for the two-dimensional Leznov lattice from the point of mathematical structure and substantial physical application. The purpose of this paper is to propose some discrete versions for the two-dimensional Leznov lattice. Until now, much attention has been paid to the problem of integrable discretizations of integrable systems. (See, e.g., [5]-[9] and references therein). It is highly nontrivial and of considerable interest to find integrable discretizations for integrable equations. Various approaches to the problem of integrable discretization are currently available. One of them is Hirota's approach [10]-[15]. Traditionally, Hirota's discretization of integrable equations is based on gauge invariance and soliton solutions. Here, we emphasize on discretizing integrable equations such that the resulting discrete bilinear equations have bilinear Bäcklund transformations. As a bonus, by doing so, we can usually derive Lax pairs for the resulting discrete equations. This method has been successfully applied to the discretization of 2+1 dimensional sinh-Gordon equation [16].

It is well known that a variety of nonlinear integrable equations share many common features, among which are so called Bäcklund transformations (BTs) and their associated nonlinear superposition formulae [17, 18, 19]. We can usually derive the nonlinear superposition formulae from the commutability of BTs. In the paper, we will show that the obtained semi-discretizations and full-discretization of the Leznov lattice do have such nice properties.

The content of the paper is organized as follows. In section 2, an y -discrete version of the Leznov lattice is found. A BT and the corresponding nonlinear superposition formula and Lax pair are shown. In section 3, an x -discrete version of the Leznov lattice is found, its associated BT, superposition principle and Lax pair are given. Furthermore, a fully discrete version of the Leznov lattice is worked out in section 3. It turns out that the resulting fully discrete version of the Leznov lattice has a BT, nonlinear superposition formula and Lax pair. In section 4, conclusion and discussions are given. Finally we list some bilinear operator identities which are used in the paper in the appendix.

2 Integrable semi-discrete version of the Leznov lattice in y -direction

By the dependent variable transformation [20]

$$\theta(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \quad p(n) = \frac{1}{2} \frac{D_x D_y f(n) \bullet f(n)}{f(n+1)f(n-1)},$$

system (1)-(2) can be transformed into the form

$$\frac{1}{2} D_y (D_x D_y f(n) \bullet f(n)) \bullet (e^{D_n} f(n) \bullet f(n)) = 2 \sinh(D_n) (e^{D_n} f(n) \bullet f(n)) \bullet f^2(n), \quad (14)$$

by introducing an auxiliary variable z and using (A31) from Appendix, we can decouple (14) into the bilinear form :

$$(D_y D_z - 2e^{D_n} + 2) f(n) \bullet f(n) = 0, \quad (15)$$

$$(D_y D_x - 2D_z e^{D_n}) f(n) \bullet f(n) = 0, \quad (16)$$

we remark that the technique of introducing auxiliary variables and then transforming multilinear equations into bilinear equations is typical in Hirota's bilinear formalism. The Hirota bilinear differential operator $D_y^m D_t^k$ and the bilinear difference operator $\exp(\delta D_n)$ are defined by [21], respectively,

$$D_y^m D_t^k a \bullet b \equiv \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^k a(y, t) b(y', t') \Big|_{y'=y, t'=t},$$

$$\exp(\delta D_n) a(n) \bullet b(n) \equiv \exp \left[\delta \left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'} \right) \right] a(n) b(n') \Big|_{n'=n} = a(n + \delta) b(n - \delta).$$

In this section, we consider an integrable semi-discretization of the Leznov lattice (1)-(2). First of all, we propose the following bilinear equations

$$\frac{1}{\delta} D_z \sinh(\delta D_y) f \bullet f = (2e^{D_n + \delta D_y} - 2e^{\delta D_y}) f \bullet f, \quad (17)$$

$$\left(\frac{1}{\delta} D_x \sinh(\delta D_y) - 2D_z e^{D_n + \delta D_y} \right) f \bullet f = 0, \quad (18)$$

By some calculations, it is shown that in the continuum limit as $\delta \rightarrow 0$, the system (17)-(18) is reduced to the Leznov lattice (15)-(16). Therefore the system (17)-(18) serves as a y-direction discrete version for (15)-(16). For simplicity we take $\delta = 1$ and rewrite variable y by m in the following discussion. In this case, the system (17)-(18) becomes

$$D_z e^{D_m} f \bullet f = (2e^{D_m + D_n} - 2e^{D_m}) f \bullet f, \quad (19)$$

$$(D_x e^{D_m} - 2D_z e^{D_m + D_n}) f \bullet f = 0. \quad (20)$$

By the dependent variable transformation

$$u_{m,n} = \ln \frac{f_{m,n+1}}{f_{m,n}}, \quad v_{m,n} = \ln \frac{f_{m+2,n}}{f_{m,n}}, \quad (21)$$

the bilinear equations (19)-(20) can be transformed into the following nonlinear form

$$u_{m+2,n} - u_{m,n} - v_{m,n+1} + v_{m,n} = 0, \quad (22)$$

$$\begin{aligned} v_{m-1,n,x} e^{u_{m-1,n-1} - u_{m+1,n-1}} - v_{m+1,n,x} e^{u_{m+1,n} - u_{m+3,n}} \\ + 4e^{u_{m+3,n+1} - u_{m+1,n-1}} - 4e^{u_{m+1,n} - u_{m-1,n-2}} = 0. \end{aligned} \quad (23)$$

In fact, from (21) we can easily derive (22). In order to obtain (23), we need to use bilinear operator identity (A32) from which it follows that

$$\frac{1}{2} \sinh(D_m) [D_x e^{D_m} f \bullet f] \bullet [e^{D_n + D_m} f \bullet f] = 2 \sinh(D_n + D_m) [e^{D_m + D_n} f \bullet f] \bullet [e^{D_m} f \bullet f] \quad (24)$$

where we have used (19) and (20). By some calculations, we can deduce (23) from (24).

Since we are looking for integrable discretization of the Leznov lattice, we need to show integrability of (19)-(20). The justification to this is via Bäcklund transformation and Lax pair. We will show that (19)-(20) is integrable in the sense of having a Bäcklund transformation and Lax pair. Concerning bilinear equations (19)-(20), we have the following result:

Proposition 1. *A Bäcklund transformation for (19) and (20) is*

$$(D_z - \lambda e^{-D_n} - \gamma) f(m, n) \bullet g(m, n) = 0, \quad (25)$$

$$(\lambda e^{D_m - \frac{1}{2} D_n} + 2e^{D_m + \frac{1}{2} D_n} + \mu e^{-\frac{1}{2} D_n - D_m}) f(m, n) \bullet g(m, n) = 0, \quad (26)$$

$$(D_x + 2\lambda e^{-D_n} - \lambda \gamma e^{-D_n} + \lambda D_z e^{-D_n} + \beta) f(m, n) \bullet g(m, n) = 0, \quad (27)$$

where λ, μ, γ and β are arbitrary constants.

Proof. Let $f(m, n)$ be a solution of Eqs. (19) and (20). If we can show that Eqs. (25)-(27) guarantee that the following two relations:

$$P_1 \equiv [D_z e^{D_m} - (2e^{D_m + D_n} - 2e^{D_m})] g(m, n) \bullet g(m, n) = 0, \quad (28)$$

$$P_2 \equiv (D_x e^{D_m} - 2D_z e^{D_m + D_n}) g(m, n) \bullet g(m, n) = 0, \quad (29)$$

hold, then Eqs. (25)-(27) form a Bäcklund transformation.

By use of (25)-(27), (A33), (A34), (A35), we have

$$\begin{aligned}
-(e^{D_m} f \bullet f) P_1 &= 2 \sinh(D_m)(D_z f \bullet g) \bullet (fg) - 4 \sinh\left(\frac{1}{2} D_n\right)(e^{\frac{D_n}{2} + D_m} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) \\
&= -2\lambda \sinh\left(\frac{D_n}{2}\right)(e^{D_m - \frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) - 4 \sinh\left(\frac{D_n}{2}\right)(e^{\frac{D_n}{2} + D_m} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) \\
&= -2 \sinh\left(\frac{D_n}{2}\right)(\lambda e^{D_m - \frac{D_n}{2}} f \bullet g + 2e^{D_m + \frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) = 0.
\end{aligned}$$

Thus we have proved that (28) holds. Similarly

$$\begin{aligned}
-(e^{D_m} f \bullet f) P_2 &= 2 \sinh(D_m)(D_x f \bullet g) \bullet (fg) - 2(D_z e^{D_n + D_m} f \bullet f)(e^{D_m} g \bullet g) + 2(e^{D_m} f \bullet f)(D_z e^{D_m + D_n} g \bullet g) \\
&\quad + 2(e^{D_m + D_n} f \bullet f)(D_z e^{D_m} g \bullet g) - 2(D_z e^{D_m} f \bullet f)(e^{D_n + D_m} g \bullet g) \\
&\quad - 2(e^{D_m + D_n} f \bullet f)(D_z e^{D_m} g \bullet g) + 2(D_z e^{D_m} f \bullet f)(e^{D_n + D_m} g \bullet g) \\
&= 2 \sinh(D_m)(D_x f \bullet g) \bullet (fg) - 4D_z \cosh\left(\frac{1}{2} D_n\right)(e^{\frac{1}{2} D_n + D_m} f \bullet g) \bullet (e^{-\frac{1}{2} D_n - D_m} f \bullet g) \\
&\quad - 4(e^{D_n + D_m} f \bullet f)[(e^{D_n + D_m} - e^{D_m})g \bullet g] + 4(e^{D_n + D_m} g \bullet g)[(e^{D_n + D_m} - e^{D_m})f \bullet f] \\
&= 2 \sinh(D_m)(D_x f \bullet g) \bullet (fg) + 2\lambda D_z \cosh\left(\frac{D_n}{2}\right)(e^{D_m - \frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) \\
&\quad + 8 \sinh\left(\frac{D_n}{2}\right)(e^{\frac{D_n}{2} + D_m} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) \\
&= 2 \sinh(D_m)(D_x f \bullet g) \bullet (fg) + 2\lambda D_z \cosh\left(\frac{D_n}{2}\right)(e^{D_m - \frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) \\
&\quad - 4 \sinh\left(\frac{D_n}{2}\right)(\lambda e^{-\frac{D_n}{2} + D_m} f \bullet g) \bullet (e^{-\frac{D_n}{2} - D_m} f \bullet g) \\
&= 2 \sinh(D_m)(D_x f \bullet g) \bullet (fg) - 4\lambda \sinh(D_m)(fg) \bullet (e^{-D_n} f \bullet g) \\
&\quad + 2\lambda \sinh(D_m)[(D_z f \bullet g) \bullet (e^{-D_n} f \bullet g) + (D_z e^{-D_n} f \bullet g) \bullet (fg)] \\
&= 2 \sinh(D_m)[(D_x + 2\lambda e^{-D_n} - \lambda \gamma e^{-D_n} + \lambda D_z e^{-D_n})f \bullet g] \bullet (fg) = 0.
\end{aligned}$$

Thus we have completed the proof of proposition 1. \square

Using (25)-(27), we can easily obtain the following solution from the trivial solution $f(m, n) = 1$:

$$g(m, n) = 1 + \exp(\eta),$$

where $\eta = pm + (\ln \frac{2}{(\lambda+2)e^{2p}-\lambda})n + [(\frac{1}{2}\lambda^2 + \lambda)e^{2p} - \frac{1}{2}\lambda^2 + \lambda]z + [(\lambda^2 + 2\lambda)e^{2p} - \lambda^2 - 2\lambda - \frac{2\lambda^2}{(\lambda+2)e^{2p}-\lambda}]x$ and $\gamma = \lambda, \mu = -\lambda - 2, \beta = \lambda^2 - 2\lambda$.

Concerning the BT (25)-(27), we have the following result:

Proposition 2. Let f_0 be a solution of the semi-discrete Leznov lattice (19)-(20). Suppose that $f_i (i = 1, 2)$ is another solution of (19)-(20) which is related to f_0 under BT (25)-(27) with parameters $(\lambda_i, \mu_i, \gamma_i, \beta_i)$ i.e., $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i, \beta_i)} f_i (i = 1, 2)$, $\lambda_1 \lambda_2 \neq 0, f_i \neq 0 (i = 0, 1, 2)$. Then f_{12} defined by

$$e^{-\frac{1}{2} D_n} f_0 \bullet f_{12} = \kappa [\lambda_1 e^{\frac{1}{2} D_n} - \lambda_2 e^{-\frac{1}{2} D_n}] f_1 \bullet f_2 \quad (30)$$

with κ being a non-zero constant, is a new solution which is related to f_1 and f_2 under the BT (25)-(27) with parameters $(\lambda_2, \mu_2, \gamma_2, \beta_2), (\lambda_1, \mu_1, \gamma_1, \beta_1)$, respectively.

As an application of the results, we can construct soliton solution of the semi-discrete Leznov lattice in y -direction. Choose, for example, $f_0 = 1, \kappa = 1/(\lambda_1 - \lambda_2)$. It can be easily verified that

$$\begin{array}{ccc}
(\lambda_1, \mu_1, \gamma_1, \beta_1) & \rightarrow & 1 + e^{\eta_1} & \xrightarrow{(\lambda_2, \mu_2, \gamma_2, \beta_2)} & F_{12} \\
1 & \rightarrow & & & \\
(\lambda_2, \mu_2, \gamma_2, \beta_2) & \rightarrow & 1 + e^{\eta_2} & \xrightarrow{(\lambda_1, \mu_1, \gamma_1, \beta_1)} &
\end{array}$$

where $e^{\eta_i} = s_i^n r_i^m e^{p_i x + q_i y + \eta_i^0}$, $p_i = 2\lambda_i(s_i - 1) + \lambda_i^2(s_i^2 - 1), q_i = \lambda_i - \lambda_i s_i, r_i = (\frac{\lambda_i + 2/s_i}{\lambda_i + 2})^{1/2}, \beta_i = -2\lambda_i - \lambda_i^2, \gamma_i = -\lambda_i, \mu_i = -2 - \lambda_i$,

$$F_{12} = 1 + \frac{\lambda_1 - \lambda_2 e^{-p_1}}{\lambda_1 - \lambda_2} e^{\eta_1} + \frac{\lambda_1 e^{-p_2} - \lambda_2}{\lambda_1 - \lambda_2} e^{\eta_2} + \frac{\lambda_1 e^{-p_2} - \lambda_2 e^{-p_1}}{\lambda_1 - \lambda_2} e^{\eta_1 + \eta_2}.$$

In Fig. 1 and Fig. 2, we show the plotting of the 2-soliton solution u_x and v_x respectively, plotted with $\lambda_1 = 2.505$, $\lambda_2 = -3.832$, $s_1 = 0.8$, $s_2 = 0.654$, $m = -4$, $z = 4$.

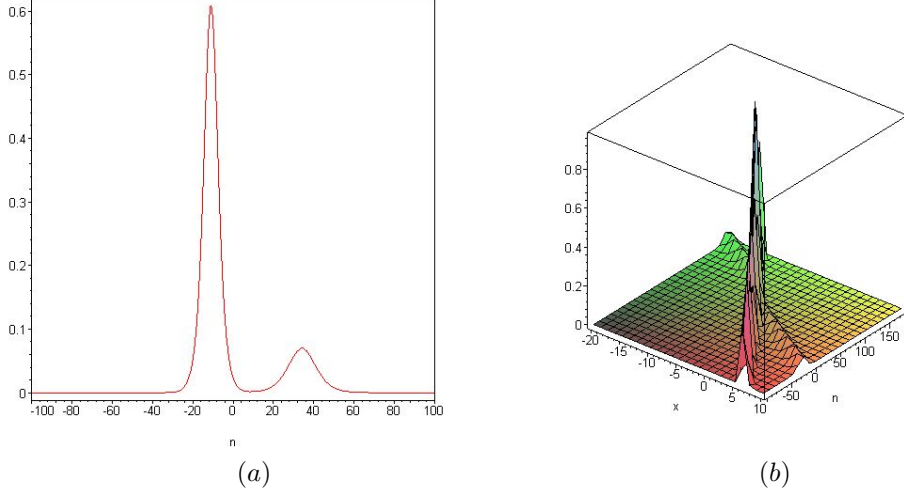


Figure 1: The 2-soliton solution: (a) $u_x(x = 0)$, (b) u_x

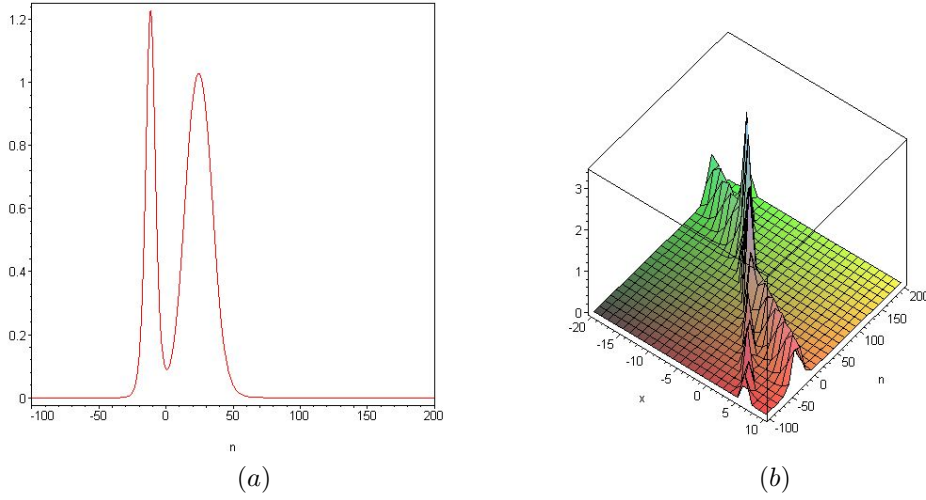


Figure 2: The 2-soliton solution: (a) $v_x(x = 0)$, (b) v_x

Starting from the bilinear BT (25)-(27), we can derive a Lax pair for the system (22)-(23). Firstly, set

$$\psi_{m,n} = f_{m,n}/g_{m,n}, \quad u_{m,n} = \ln \frac{g_{m,n+1}}{g_{m,n}}, \quad v_{m,n} = \ln \frac{g_{m+2,n}}{g_{m,n}}$$

in (25)-(27). Then from the bilinear BT (25)-(27) and after some calculations, we can obtain the following Lax pair for (22)-(23):

$$\lambda \psi_{m+1,n} e^{u_{m-1,n} - u_{m+1,n}} + 2\psi_{m+1,n+1} + \mu \psi_{m-1,n} = 0, \quad (31)$$

$$\begin{aligned} \psi_{m,n,x} + \lambda^2 \psi_{m,n-2} e^{u_{m,n} - u_{m,n-2}} - \frac{1}{2} \lambda \psi_{m,n-1} v_{m,n,x} e^{u_{m,n} - u_{m+2,n}} \\ + 2\lambda \psi_{m,n-1} e^{u_{m+2,n+1} - u_{m,n-1}} + \beta \psi_{m,n} = 0. \end{aligned} \quad (32)$$

By differentiating eq. (31) with respect to x and by use of (32) to eliminate $\psi_{m+1,x}, \psi_{m+1,n+1,x}, \psi_{m-1,x}$, also by use of (31) to express $\psi_{m-1,i}$ by $\psi_{m+1,i}$ and $\psi_{m+1,i+1}$ ($i = n-2, n-1, n$), we can find that the coefficient of $\lambda^2 \psi_{m+1,n-1}$ is just (23). Together with the coefficient of $\lambda \psi_{m+1,n}$ we can obtain (22). So the compatibility condition of (31) and (32) yields the semi-discrete system (22)-(23).

3 Integrable semi-discrete version of the Leznov lattice in x-direction

Similar to section 2, we first propose the following bilinear equations:

$$D_y D_z f \bullet f = (2e^{D_n} - 2)f \bullet f, \quad (33)$$

$$\frac{1}{\epsilon} \sinh(\epsilon D_x) D_y f \bullet f - 2D_z e^{D_n + \epsilon D_x} f \bullet f = 0. \quad (34)$$

By some calculations, it is shown that in the continuum limit as $\epsilon \rightarrow 0$, the system (33)-(34) is reduced to the Leznov lattice (15)-(16). Therefore the system (33)-(34) serves as a x-direction discrete version for (15)-(16). For simplicity we take $\epsilon = 1$ and rewrite variable x by k in the following discussion. In this case, the system (33)-(34) becomes

$$D_y D_z f \bullet f = (2e^{D_n} - 2)f \bullet f, \quad (35)$$

$$(D_y e^{D_k} - 2D_z e^{D_n + D_k})f \bullet f = 0. \quad (36)$$

Through the dependent variable transformation

$$u_{n,k} = \ln \frac{f_{n+1,k}}{f_{n,k}}, \quad v_{n,k} = \ln \frac{f_{n,k+1}}{f_{n,k-1}}, \quad (37)$$

the bilinear equations (35)-(36) can be transformed into the following nonlinear form

$$v_{n+1,k} - v_{n,k} + u_{n,k-1} - u_{n,k+1} = 0, \quad (38)$$

$$2e^{u_{n+1,k+1} - u_{n-1,k-1}} - 2e^{u_{n+2,k+1} - u_{n,k-1}} + v_{n+1,k,y} + v_{n+1,k,y}(u_{n,k-1,y} - u_{n+1,k+1,y}) = 0. \quad (39)$$

In fact, from (37) we can easily derive (38). In order to obtain (39), we need to use bilinear operator identity (A38) from which it follows that

$$\frac{1}{2} D_y [D_y e^{D_k} f \bullet f] \bullet [e^{D_n + D_k} f \bullet f] = 2 \sinh(D_n + D_k) [e^{D_n} f \bullet f] \bullet f^2 \quad (40)$$

where we have used (35) and (36). By some calculations, we can deduce (39) from (40).

Concerning the bilinear equations (35)-(36), we have the following results:

Proposition 3. *A bilinear BT for equation (35) and (36) is*

$$(D_y + \lambda^{-1} e^{-D_n} + \mu) f(n, k) \bullet g(n, k) = 0, \quad (41)$$

$$(D_z e^{-\frac{D_n}{2}} - \lambda e^{\frac{D_n}{2}} + \gamma e^{-\frac{D_n}{2}}) f(n, k) \bullet g(n, k) = 0, \quad (42)$$

$$(\lambda^{-1} e^{-\frac{D_n}{2} + D_k} - 2D_z e^{\frac{D_n}{2} + D_k} - 2\gamma e^{\frac{D_n}{2} + D_k} + \beta e^{-\frac{D_n}{2} - D_k}) f(n, k) \bullet g(n, k) = 0. \quad (43)$$

where λ, μ, γ and β are arbitrary constants.

Proof. Let $f(n, k)$ be a solution of Eqs. (35) and (36). If we can show that Eqs. (41)-(43) guarantee that the following two relations:

$$P_1 \equiv [D_y D_z - (2e^{D_n} - 2)]g(n, k) \bullet g(n, k) = 0, \quad (44)$$

$$P_2 \equiv (D_y e^{D_k} - 2D_z e^{D_n + D_k})g(n, k) \bullet g(n, k) = 0, \quad (45)$$

hold, then Eqs. (41)-(43) form a *Bäcklund* transformation. In analogy with the proof already given in [20], we know that $P_1 = 0$ holds. Thus it suffices to show that $P_2 = 0$. In this regard, by using (A33), (A39),

(A40), we have

$$\begin{aligned}
-P_2[e^{D_k} f \bullet f] &= 2 \sinh(D_k)(D_y f \bullet g) \bullet (fg) - 4 \sinh\left(\frac{D_n}{2}\right)(D_z e^{\frac{D_n}{2}+D_k} f \bullet g) \bullet (e^{-\frac{D_n}{2}-D_k} f \bullet g) \\
&\quad + 4 \sinh\left(\frac{D_n}{2} + D_k\right)(e^{\frac{D_n}{2}} f \bullet g) \bullet (D_z e^{-\frac{D_n}{2}} f \bullet g) \\
&= -2\lambda^{-1} \sinh(D_k)(e^{-D_n} f \bullet g) \bullet (fg) - 4 \sinh\left(\frac{D_n}{2}\right)(D_z e^{\frac{D_n}{2}+D_k} f \bullet g) \bullet (e^{-\frac{D_n}{2}-D_k} f \bullet g) \\
&\quad - 4\gamma \sinh\left(\frac{D_n}{2} + D_k\right)(e^{\frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2}} f \bullet g) \\
&= 2\lambda^{-1} \sinh\left(\frac{D_n}{2}\right)(e^{-\frac{D_n}{2}+D_k} f \bullet g) \bullet (e^{-\frac{D_n}{2}-D_k} f \bullet g) - 4 \sinh\left(\frac{D_n}{2}\right)(D_z e^{\frac{D_n}{2}+D_k} f \bullet g) \bullet (e^{-\frac{D_n}{2}-D_k} f \bullet g) \\
&\quad - 4\gamma \sinh\left(\frac{D_n}{2}\right)(e^{\frac{D_n}{2}+D_k} f \bullet g) \bullet (e^{-\frac{D_n}{2}-D_k} f \bullet g) \\
&= 2 \sinh\left(\frac{D_n}{2}\right)[(\lambda^{-1} e^{-\frac{D_n}{2}+D_k} - 2D_z e^{\frac{D_n}{2}+D_k} - 2\gamma e^{\frac{D_n}{2}+D_k}) f \bullet g] \bullet (e^{-\frac{D_n}{2}-D_k} f \bullet g) = 0.
\end{aligned}$$

Thus we have completed the proof of Proposition 3. \square

Using (41)-(43), we can easily obtain the following solution from the trivial solution $f(m, n) = 1$:

$$g(m, n) = 1 + \exp(\eta),$$

where $\eta = pn + \left(\frac{1}{2} \ln \frac{\lambda^{-1} - 2\lambda e^{-2p}}{\lambda^{-1} - 2\lambda}\right)k + (\lambda^{-1} e^p - \lambda^{-1})y + (\lambda - \lambda e^{-p})z$ and $\gamma = \lambda, \mu = -\lambda - 2, \beta = \lambda^2 - 2\lambda$. Concerning the BT (41)-(43), we have the following result:

Proposition 4. *Let f_0 be a solution of the semi-discrete Leznov lattice (35)-(36). Suppose that $f_i (i = 1, 2)$ is another solution of (35)-(36) which is related to f_0 under BT (41)-(43) with parameters $(\lambda_i, \mu_i, \gamma_i, \beta_i)$ i.e., $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i, \beta_i)} f_i (i = 1, 2)$, $\lambda_1 \lambda_2 \neq 0, f_i \neq 0 (i = 0, 1, 2)$. Then f_{12} defined by*

$$e^{-\frac{1}{2}D_n} f_0 \bullet f_{12} = \frac{1}{c} [\lambda_2 e^{\frac{1}{2}D_n} - \lambda_1 e^{-\frac{1}{2}D_n}] f_1 \bullet f_2 \quad (c \text{ is a nonzero constant}), \quad (46)$$

is a new solution which is related to f_1 and f_2 under the BT (41)-(43) with parameters $(\lambda_2, \mu_2, \gamma_2, \beta_2), (\lambda_1, \mu_1, \gamma_1, \beta_1)$, respectively.

Starting from the bilinear BT (41)-(43), we can derive a Lax pair for the system (38)-(39). Firstly, set

$$\psi_{n,k} = f_{n,k}/g_{n,k}, \quad u_{n,k} = \ln \frac{g_{n+1,k}}{g_{n,k}}, \quad v_{n,k} = \ln \frac{g_{n,k+1}}{g_{n,k-1}}$$

in (41)-(43). Then from the bilinear BT (41)-(43) and after some calculations, we can obtain the following Lax pair for (38)-(39):

$$\lambda^{-1} \psi_{n,k+1} e^{u_{n,k-1} - u_{n,k+1}} - 2\lambda \psi_{n+2,k+1} - v_{n+1,k,y} \psi_{n+1,k+1} e^{u_{n,k-1} - u_{n+1,k+1}} + \beta \psi_{n,k-1} = 0, \quad (47)$$

$$\psi_{n,k,y} + \lambda^{-1} \psi_{n-1,k} e^{u_{n,k} - u_{n-1,k}} + \mu \psi_{n,k} = 0. \quad (48)$$

Similar to the deduction of compatibility in the above section, by differentiating eq. (47) with y and substitute (48) into the differential expression to eliminate $\psi_{k+1,y}, \psi_{n+2,k+1,y}, \psi_{n+1,k+1,y}, \psi_{k-1,y}$, it can be shown that the compatibility condition of (47)-(48) generates the system (38)-(39).

4 Integrable fully-discrete version of the Leznov lattice

First of all, we propose the following bilinear equations

$$\frac{1}{\delta} \sinh(\delta D_y) D_z f \bullet f = (2e^{D_n + \delta D_y} - 2e^{\delta D_y}) f \bullet f, \quad (49)$$

$$\left[\frac{1}{\delta \epsilon} \sinh(\delta D_y) \sinh(\epsilon D_x) - 2D_z e^{D_n + \delta D_y + \epsilon D_x} \right] f \bullet f = 0. \quad (50)$$

In the continuum limit as $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$, the system (49)-(50) is reduced to the Leznov lattice (15)-(16). Therefore the system (49)-(50) serves as a full-discrete version for (15)-(16). For simplicity we take

$\epsilon = 1, \delta = 1$ and rewrite variable y by m and x by k in the following discussion. In this case the system (49)-(50) becomes

$$D_z e^{D_m} f \bullet f = (2e^{D_n+D_m} - 2e^{D_m}) f \bullet f, \quad (51)$$

$$[\sinh(D_m) \sinh(D_k) - 2D_z e^{D_n+D_m+D_k}] f \bullet f = 0. \quad (52)$$

Through the independent variable transformation

$$u_{m,n,k} = \ln \frac{f_{m,n+1,k}}{f_{m,n,k}}, v_{m,n,k} = \ln \frac{f_{m,n,k+2}}{f_{m,n,k}}, \quad (53)$$

eqs. (51) and (52) can be transformed into the nonlinear form

$$\begin{aligned} & e^{u_{m-1,n-1,k+1}-u_{m+1,n,k+1}+v_{m-1,n-1,k-1}-v_{m+1,n,k-1}} - e^{u_{m-1,n-1,k-1}-u_{m+1,n,k+1}} \\ & + e^{u_{m-3,n-1,k-1}-u_{m-1,n,k+1}} - e^{u_{m-3,n-1,k+1}-u_{m-1,n,k+1}+v_{m-3,n-1,k-1}-v_{m-1,n,k-1}} \\ & + 8e^{u_{m+1,n+1,k+1}-u_{m-1,n,k+1}} - 8e^{u_{m-1,n-1,k-1}-u_{m-3,n-2,k-1}} = 0, \end{aligned} \quad (54)$$

$$u_{m,n,k+2} - u_{m,n,k} + v_{m,n,k} - v_{m,n+1,k} = 0. \quad (55)$$

In fact, from (53) we can easily derive (55). In order to obtain (54), we need to use bilinear operator identity (A41) from which it follows that

$$\frac{1}{2} \sinh(D_m) [\sinh(D_m) \sinh(D_k) f \bullet f] \bullet [e^{D_m+D_n+D_k} f \bullet f] = 2 \sinh(D_m + D_n + D_k) (e^{D_m+D_n} f \bullet f) \bullet (e^{D_m} f \bullet f) \quad (56)$$

where we have used (51) and (52). By some calculations, we can deduce (54) from (56).

Concerning the bilinear equations (51)-(52), we have the following results:

Proposition 5. *A Bäcklund transformation for the fully discrete Leznov lattice (51)-(52) is*

$$(D_z e^{\frac{D_n}{2}} - \lambda e^{-\frac{D_n}{2}} - \gamma e^{\frac{D_n}{2}}) f \bullet g = 0, \quad (57)$$

$$(\lambda e^{D_m} + 2e^{D_m+D_n} + \mu e^{-D_m}) f \bullet g = 0, \quad (58)$$

$$(D_z e^{-\frac{D_n}{2}-D_k-D_m} - \gamma e^{-\frac{D_n}{2}-D_k-D_m} + \frac{1}{2\mu} e^{D_m+\frac{D_n}{2}-D_k} + \beta e^{\frac{D_n}{2}+D_k+D_m}) f \bullet g = 0. \quad (59)$$

where λ, μ, γ and β are arbitrary constants.

Proof. Let $f(n)$ be a solution of Eqs. (51) and (52). What we need to prove is that the function g satisfying (57)-(59) is another solution of Eqs. (51) and (52), i.e.,

$$P_1 \equiv [D_z e^{D_m} - (2e^{D_n+D_m} - 2e^{D_m})] g \bullet g = 0, \quad (60)$$

$$P_2 \equiv [\sinh(D_m) \sinh(D_k) - 2D_z e^{D_n+D_m+D_k}] g \bullet g = 0. \quad (61)$$

In the section 2, we have proved that (60) holds. Thus it suffices to show that $P_2 = 0$. In this regards, by (A42)-(A43), we have

$$\begin{aligned} & -P_2 [e^{D_m+D_k} f \bullet f] = [(\sinh(D_m) \sinh(D_k) - 2D_z e^{D_n+D_m+D_k}) f \bullet f] [e^{D_m+D_k} g \bullet g] \\ & - [(\sinh(D_m) \sinh(D_k) - 2D_z e^{D_n+D_m+D_k}) g \bullet g] [e^{D_m+D_k} f \bullet f] \\ & = \sinh(D_k) (e^{D_m} f \bullet g) \bullet (e^{-D_m} f \bullet g) - 4 \sinh\left(\frac{D_n}{2} + D_k + D_m\right) (D_z e^{\frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2}} f \bullet g) \\ & + 4 \sinh\left(\frac{D_n}{2}\right) (e^{\frac{D_n}{2}+D_k+D_m} f \bullet g) \bullet (D_z e^{-\frac{D_n}{2}-D_k-D_m} f \bullet g) \\ & = -\frac{\mu}{2} \sinh(D_k) (e^{D_m} f \bullet g) \bullet (e^{D_m+D_n} f \bullet g) - 4\gamma \sinh\left(\frac{D_n}{2} + D_k + D_m\right) (e^{\frac{D_n}{2}} f \bullet g) \bullet (e^{-\frac{D_n}{2}} f \bullet g) \\ & + 4 \sinh\left(\frac{D_n}{2}\right) (e^{\frac{D_n}{2}+D_k+D_m} f \bullet g) \bullet (D_z e^{-\frac{D_n}{2}-D_k-D_m} f \bullet g) \\ & = \frac{\mu}{2} \sinh\left(\frac{D_n}{2}\right) (e^{\frac{D_n}{2}+D_m+D_k} f \bullet g) \bullet (e^{D_m+\frac{D_n}{2}-D_k} f \bullet g) - 4\gamma \sinh\left(\frac{D_n}{2}\right) (e^{\frac{D_n}{2}+D_k+D_m} f \bullet g) \bullet (e^{-\frac{D_n}{2}-D_k-D_m} f \bullet g) \\ & + 4 \sinh\left(\frac{D_n}{2}\right) (e^{\frac{D_n}{2}+D_k+D_m} f \bullet g) \bullet (D_z e^{-\frac{D_n}{2}-D_k-D_m} f \bullet g) \\ & = 2 \sinh\left(\frac{D_n}{2}\right) (e^{\frac{D_n}{2}+D_m+D_k} f \bullet g) \bullet [(\mu^{-1} e^{D_m+\frac{D_n}{2}-D_k} - 2\gamma e^{-\frac{D_n}{2}-D_k-D_m} + 2D_z e^{-\frac{D_n}{2}-D_k-D_m}) f \bullet g] = 0. \end{aligned}$$

Thus we have completed the proof of Proposition 5. \square

Using the BT (57)-(59), we can obtain the following solution from the trivial solution $f(m, n, k) = 1$:

$$g(m, n, k) = 1 + \exp(\eta)$$

where $\eta = \frac{1}{2}(\ln \frac{\lambda+2e^{-q}}{\lambda+2})m + qn + \frac{1}{2}(\ln \frac{2\lambda^2+4\lambda-1}{\lambda^2e^{2q}+4\lambda e^q-1})k + (\lambda - \lambda e^q)z$ and $\mu = -2 - \lambda, \beta = -\lambda + \frac{1}{4+2\lambda}, \gamma = -\lambda$.

Proposition 6. *Let f_0 be a solution of the fully discrete Leznov lattice (51)-(52). Suppose that $f_i (i = 1, 2)$ is another solution of (51)-(52) which is related to f_0 under BT (57)-(59) with parameters $(\lambda_i, \mu_i, \gamma_i, \beta_i)$ i.e., $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i, \beta_i)} f_i (i = 1, 2)$, $\lambda_1 \lambda_2 \neq 0, f_i \neq 0 (i = 0, 1, 2)$. Then f_{12} defined by*

$$(\lambda_1 e^{\frac{1}{2}D_n} - \lambda_2 e^{-\frac{1}{2}D_n})f_1 \bullet f_2 = k e^{\frac{1}{2}D_n} f_0 \bullet f_{12} \quad (k \text{ is a nonzero constant}), \quad (62)$$

is a new solution which is related to f_1 and f_2 under the BT (57)-(59) with parameters $(\lambda_2, \mu_2, \gamma_2, \beta_2), (\lambda_1, \mu_1, \gamma_1, \beta_1)$, respectively.

In the following, we derive a Lax pair for the fully discrete Leznov equation (51)-(52). Let

$$\psi = \frac{f}{g}, \quad u = \ln \frac{g_{n+1}}{g}, \quad v = \ln \frac{g_{k+2}}{g}.$$

Then from eqs. (57)-(59) we can obtain

$$\begin{aligned} & -\lambda^2 \psi_{n-1, k-1} - (2\lambda C_{m-1, n-1, k-1} + \frac{\lambda}{4} A_{m-1, n, k}) \psi_{k-1} \\ & + (\frac{1}{2} B_{m-1, n, k} - \frac{1}{2} A_{m-1, n, k} C_{m-1, n, k-1}) \psi_{n+1, k-1} + \mu \beta \psi_{n+1, k+1} = 0, \end{aligned} \quad (63)$$

$$\lambda \psi_{m+1} + \mu \psi_{m-1} + 2C_{m, n, k} \psi_{m+1, n+1} = 0. \quad (64)$$

In the above, we have denoted

$$\begin{aligned} A_{m, n, k} & \equiv e^{u_{m-1, n-1, k-1} + v_{m-1, n, k-1} - u_{m+1, n, k-1} - v_{m+1, n+1, k-1}} - e^{u_{m-1, n-1, k-1} - u_{m+1, n, k+1}}, \\ B_{m, n, k} & \equiv e^{v_{m-1, n, k-1} - v_{m+1, n+1, k-1}}, \quad C_{m, n, k} \equiv e^{u_{m+1, n, k} - u_{m-1, n-1, k}}, \end{aligned}$$

for simplicity. In order to derive Lax pair for (54)-(55), we first shift k to $k+1$ in eq.(64), then substitute those ψ with index $k+1$ by $k-1$, $m-1$ by $m+1$ by use of (63) and (64). Through some calculations we can see that the compatibility condition of (63)-(64) generates eqs. (54)-(55). So they constitute the Lax pair for (54)-(55).

5 Conclusion and discussions

In this paper, we have presented two semi-discrete integrable versions and one fully discrete integrable version for the Leznov lattice. Corresponding BTs, nonlinear superposition formulae and Lax pairs are derived. Based on these obtained results, it is natural to further consider some other integrable properties such as determinant structures of N-soliton solutions for these semi-discrete and fully discrete integrable versions of the Leznov lattice. Besides, it would be interesting to work out their pfaffian versions for these semi-discrete and fully discrete Leznov lattice. Finally it is noted that in the paper the semi-discrete and fully discrete forms are always compared with bilinear equations (15)-(16). It is natural to inquire what are discrete analogues of (1)-(2) and what are the discretizations of p and θ . In order to answer these, let us first consider y -directional discrete version of the Leznov lattice (17)-(18). In this case, we have the following equation, by use of (A32) with $D_m = \delta D_y$,

$$\frac{1}{2\delta^2} \sinh(\delta D_y) [D_x \sinh(\delta D_y) f \bullet f] \bullet [e^{D_n + \delta D_y} f \bullet f] = 2 \sinh(D_n + \delta D_y) [e^{\delta D_y + D_n} f \bullet f] \bullet [e^{\delta D_y} f \bullet f]. \quad (65)$$

By the dependent variable transformation

$$\theta(n, x, y) = \frac{f(n+1, x, y + \delta) f(n-1, x, y - \delta)}{f(n, x, y + \delta) f(n, x, y - \delta)}, \quad p(n, x, y) = \frac{1}{2\delta} \frac{D_x \sinh(\delta D_y) f \bullet f}{f(n+1, x, y + \delta) f(n-1, x, y - \delta)} \quad (66)$$

From (65) and (66), we can easily derive y -directional discretization of the Leznov lattice (1)-(2)

$$\begin{aligned} \frac{1}{2\delta}(p(n, x, y + \delta) - p(n, x, y - \delta)) &= \theta(n + 1, x, y + \delta) - \theta(n - 1, x, y - \delta) \\ \frac{\partial}{\partial x} \frac{1}{\delta} \sinh(\delta \partial_y) \ln \theta(n, x, y) &= \theta(n + 1, x, y + \delta)p(n + 1, x, y + \delta) - \theta(n, x, y + \delta)p(n, x, y + \delta) \\ &\quad - \theta(n, x, y - \delta)p(n, x, y - \delta) + \theta(n - 1, x, y - \delta)p(n - 1, x, y - \delta). \end{aligned}$$

We now turn to consider x -directional discrete version of the Leznov lattice (33)-(34). In this case, by the following dependent variable transformation

$$\begin{aligned} \theta(n, x, y) &= \frac{f(n + 1, x, y)f(n - 1, x, y)}{f(n, x, y)f(n, x, y)}, \\ p(n, x, y) &= \frac{1}{2\epsilon} \frac{D_y \sinh(\epsilon D_x) f \bullet f}{f(n + 1, x + \epsilon, y)f(n - 1, x - \epsilon, y)}, \quad q(n, x, y) = \frac{f(n + 1, x, y)}{f(n, x, y)} \end{aligned}$$

and by use of (A38) with $D_n = \epsilon D_x$, we have from (33)-(34) the following x -directional discrete version of the Leznov lattice (1)-(2)

$$p_y(n, x, y) = \theta(n + 1, x + \epsilon, y) - \theta(n - 1, x - \epsilon, y) \quad (67)$$

$$\theta(n, x, y) = \frac{q(n, x, y)}{q(n - 1, x, y)} \quad (68)$$

$$\begin{aligned} \frac{\partial}{\partial y} \frac{1}{\epsilon} \sinh(\epsilon \partial_x) \ln \theta(n, x, y) &= p(n + 1, x, y) \frac{q(n + 1, x + \epsilon, y)}{q(n, x - \epsilon, y)} \\ &\quad - 2p(n, x, y) \frac{q(n, x + \epsilon, y)}{q(n - 1, x - \epsilon, y)} + p(n - 1, x, y) \frac{q(n - 1, x + \epsilon, y)}{q(n - 2, x - \epsilon, y)}. \end{aligned} \quad (69)$$

Similarly we can also further consider fully discrete version of the Leznov lattice (1)-(2) by the dependent variable transformation

$$\begin{aligned} \theta(n, x, y) &= \frac{f(n + 1, x, y + \delta)f(n - 1, x, y - \delta)}{f(n, x, y + \delta)f(n, x, y - \delta)}, \quad p(n, x, y) = \frac{1}{2\epsilon\delta} \frac{\sinh(\epsilon D_x) \sinh(\delta D_y) f \bullet f}{f(n + 1, x + \epsilon, y + \delta)f(n - 1, x - \epsilon, y - \delta)} \\ r_\epsilon(n, x, y) &= \frac{f(n + 1, x + \epsilon, y)f(n, x - \epsilon, y)}{f(n + 1, x - \epsilon, y)f(n, x + \epsilon, y)} \end{aligned}$$

and by use of (A41) with $D_k = \epsilon D_x$ and $D_m = \delta D_y$. In fact, from (49)-(50), we have

$$\begin{aligned} \frac{1}{2\delta}[p(n, x, y + \delta) - p(n, x, y - \delta)] &= \theta(n + 1, x + \epsilon, y + \delta) - \theta(n - 1, x - \epsilon, y - \delta) \\ 4\epsilon\delta[\theta(n, x - \epsilon, y)p(n, x, y) - \theta(n + 1, x + \epsilon, y)p(n + 1, x, y)] &= \frac{1}{r_\epsilon(n, x, y + \delta)} - \frac{1}{r_\epsilon(n, x, y - \delta)} \\ \frac{1}{\epsilon} \frac{1}{\delta} \sinh(\epsilon \partial_x) \sinh(\delta \partial_y) \ln \theta(n, x, y) &= \frac{1}{4\epsilon\delta} [\ln r_\epsilon(n, x, y + 2\delta) - \ln r_\epsilon(n, x, y) - \ln r_\epsilon(n - 1, x, y) + \ln r_\epsilon(n - 1, x, y - 2\delta)]. \end{aligned}$$

It can be easily verified that when $\delta \rightarrow 0$, the above system becomes (67)-(69).

Acknowledgements

The authors would like to thank the referee for valuable comments. This work was supported by the National Natural Science Foundation of China (grant no. 10771207), the knowledge innovation program of the Institute of Computational Math., AMSS and State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences.

Appendix A. Proofs of propositions 2, 4, 6

A.1 Proof of Proposition 2:

Analogous to the deduction in [22] and [23], we can show that

$$-D_z f_1 \bullet f_2 + (\gamma_2 - \gamma_1) f_1 f_2 - \frac{1}{\kappa} e^{-D_n} f_0 \bullet f_{12} = 0, \quad (\text{A1})$$

$$(D_z - \lambda_2 e^{D_n} - \gamma_2) f_1 \bullet f_{12} = 0, \quad (\text{A2})$$

$$(D_z - \lambda_1 e^{D_n} - \gamma_1) f_2 \bullet f_{12} = 0. \quad (\text{A3})$$

$$(\lambda_2 e^{D_m - \frac{1}{2} D_n} + 2e^{D_m + \frac{1}{2} D_n} + \mu_2 e^{-\frac{1}{2} D_n - D_m}) f_1 \bullet f_{12} = 0, \quad (\text{A4})$$

$$(\lambda_1 e^{D_m - \frac{1}{2} D_n} + 2e^{D_m + \frac{1}{2} D_n} + \mu_1 e^{-\frac{1}{2} D_n - D_m}) f_2 \bullet f_{12} = 0, \quad (\text{A5})$$

Thus, in order to prove Proposition 2, it suffices to show that

$$(D_x + 2\lambda_2 e^{-D_n} - \lambda_2 \gamma_2 e^{-D_n} + \lambda_2 D_z e^{-D_n} + \beta_2) f_1 \bullet f_{12} = 0, \quad (\text{A6})$$

$$(D_x + 2\lambda_1 e^{-D_n} - \lambda_1 \gamma_1 e^{-D_n} + \lambda_1 D_z e^{-D_n} + \beta_1) f_2 \bullet f_{12} = 0. \quad (\text{A7})$$

Since f_1 and f_2 are two solutions of (19), we have, by using (30), (A34)-(A37) and $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i, \beta_i)}$ f_i ($i = 1, 2$), that

$$\begin{aligned} 0 &= [(\lambda_1 e^{D_m - \frac{D_n}{2}} + 2e^{D_m + \frac{D_n}{2}} + \mu_1 e^{-\frac{D_n}{2} - D_m}) f_0 \bullet f_1] [e^{-D_m - \frac{D_n}{2}} f_2 \bullet f_0] \\ &\quad - [(\lambda_2 e^{D_m - \frac{D_n}{2}} + 2e^{D_m + \frac{D_n}{2}} + \mu_2 e^{-\frac{D_n}{2} - D_m}) f_0 \bullet f_2] [e^{-D_m - \frac{D_n}{2}} f_1 \bullet f_0] \\ &= e^{D_m} (e^{-\frac{D_n}{2}} f_0 \bullet f_0) \bullet [(\lambda_1 e^{\frac{D_n}{2}} - \lambda_2 e^{-\frac{D_n}{2}}) f_1 \bullet f_2] + (e^{-D_m - \frac{D_n}{2}} f_0 \bullet f_0) [(\mu_1 e^{D_m + \frac{D_n}{2}} - \mu_2 e^{-D_m - \frac{D_n}{2}}) f_1 \bullet f_2] \\ &= (e^{-D_m - \frac{D_n}{2}} f_0 \bullet f_0) \left[\frac{1}{\kappa} e^{D_m - \frac{D_n}{2}} f_0 \bullet f_{12} + \mu_1 e^{D_m + \frac{D_n}{2}} f_1 \bullet f_2 - \mu_2 e^{-D_m - \frac{D_n}{2}} f_1 \bullet f_2 \right], \end{aligned}$$

so we have

$$\frac{1}{\kappa} e^{D_m - \frac{D_n}{2}} f_0 \bullet f_{12} = (\mu_2 e^{-D_m - \frac{D_n}{2}} - \mu_1 e^{D_m + \frac{D_n}{2}}) f_1 \bullet f_2. \quad (\text{A8})$$

Similarly,

$$\begin{aligned} 0 &= [(e^{D_m - \frac{D_n}{2}} + \frac{2}{\lambda_1} e^{D_m + \frac{D_n}{2}} + \frac{\mu_1}{\lambda_1} e^{-\frac{D_n}{2} - D_m}) f_0 \bullet f_1] [e^{-D_m + \frac{D_n}{2}} f_2 \bullet f_0] \\ &\quad - [(e^{D_m - \frac{D_n}{2}} + \frac{2}{\lambda_2} e^{D_m + \frac{D_n}{2}} + \frac{\mu_2}{\lambda_2} e^{-\frac{D_n}{2} - D_m}) f_0 \bullet f_2] [e^{-D_m + \frac{D_n}{2}} f_1 \bullet f_0] \\ &= \frac{2}{\lambda_1 \lambda_2} e^{D_m} (e^{\frac{D_n}{2}} f_0 \bullet f_0) \bullet [(\lambda_2 e^{-\frac{D_n}{2}} - \lambda_1 e^{\frac{D_n}{2}}) f_1 \bullet f_2] + e^{\frac{D_n}{2}} (e^{-D_m} f_0 \bullet f_0) \bullet \left[\left(\frac{\mu_1}{\lambda_1} e^{D_m} - \frac{\mu_2}{\lambda_2} e^{-D_m} \right) f_1 \bullet f_2 \right] \\ &= e^{-\frac{D_n}{2}} (e^{-D_m} f_0 \bullet f_0) \bullet \left[-\frac{2}{\lambda_1 \lambda_2 \kappa} e^{D_m} f_0 \bullet f_{12} + \frac{\mu_1}{\lambda_1} e^{D_m} f_1 \bullet f_2 - \frac{\mu_2}{\lambda_2} e^{-D_m} f_1 \bullet f_2 \right], \end{aligned}$$

which means that

$$\frac{2}{\kappa} e^{D_m} f_0 \bullet f_{12} = (\mu_1 \lambda_2 e^{D_m} - \mu_2 \lambda_1 e^{-D_m}) f_1 \bullet f_2. \quad (\text{A9})$$

Since f_1 and f_2 are two solutions of (20), we have, by use of (30),(A8), (A9),that

$$\begin{aligned}
0 &= [(D_x e^{D_m} - 2D_z e^{D_n+D_m})f_1 \bullet f_1]e^{D_m} f_2 \bullet f_2 - [(D_x e^{D_m} - 2D_z e^{D_n+D_m})f_2 \bullet f_2]e^{D_m} f_1 \bullet f_1 \\
&= D_x(e^{D_m} f_1 \bullet f_2) \bullet (e^{-D_m} f_1 \bullet f_2) - 4D_z \cosh\left(\frac{D_n}{2}\right)(e^{D_m+\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad + 8 \sinh\left(\frac{D_n}{2}\right)(e^{D_m+\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&= \frac{1}{\mu_1 \lambda_2} D_x[(\mu_1 \lambda_2 e^{D_m} - \mu_2 \lambda_1 e^{-D_m})f_1 \bullet f_2] \bullet (e^{-D_m} f_1 \bullet f_2) \\
&\quad - \frac{4}{\mu_1} D_z \cosh\left(\frac{D_n}{2}\right)[(\mu_1 e^{D_m+\frac{D_n}{2}} - \mu_2 e^{-D_m-\frac{D_n}{2}})f_1 \bullet f_2] \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad + \frac{8}{\mu_1} \sinh\left(\frac{D_n}{2}\right)[(\mu_1 e^{D_m+\frac{D_n}{2}} - \mu_2 e^{-D_m-\frac{D_n}{2}})f_1 \bullet f_2] \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&= \frac{2}{\mu_1 \lambda_2 k} D_x(e^{D_m} f_0 \bullet f_{12}) \bullet (e^{-D_m} f_1 \bullet f_2) + \frac{4}{\kappa \mu_1} D_z \cosh\left(\frac{D_n}{2}\right)(e^{D_m+\frac{D_n}{2}} f_0 \bullet f_{12}) \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad - \frac{8}{k \mu_1} \sinh\left(\frac{D_n}{2}\right)(e^{D_m+\frac{D_n}{2}} f_0 \bullet f_{12}) \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&= \frac{2}{\mu_1 \lambda_2 \kappa} D_x(e^{D_m} f_0 \bullet f_{12}) \bullet (e^{-D_m} f_1 \bullet f_2) + \frac{4}{\kappa \mu_1} e^{D_m} [(D_z f_0 \bullet f_2) \bullet (e^{-D_n} f_1 \bullet f_{12}) + (f_0 f_2) \bullet (-D_z e^{-D_n} f_1 \bullet f_{12}) \\
&\quad + (D_z e^{-D_n} f_0 \bullet f_2) \bullet (f_1 f_{12}) - (e^{D_n} f_0 \bullet f_2) \bullet (D_z f_1 \bullet f_{12})] - \frac{8}{\kappa \mu_1} \sinh\left(\frac{D_n}{2}\right)(e^{D_m+\frac{D_n}{2}} f_0 \bullet f_{12}) \bullet (e^{-D_m-\frac{D_n}{2}} f_1 \bullet f_2) \\
&= \frac{2}{\kappa \mu_1} e^{D_m} (f_0 f_2) \bullet [(\gamma_2 e^{-D_n} - D_z e^{-D_n} - 2e^{-D_n} - \frac{1}{\lambda_2} D_x - \frac{\beta_2}{\lambda_2})f_1 \bullet f_{12}]
\end{aligned}$$

which implies that (A6) holds. Similarly, we can show that (A7) also holds. Therefore, we have completed the proof of Proposition 2.

A.2 Proof of Proposition 4:

Analogous to the deduction in [22] and [23], we can show that

$$-D_y f_1 \bullet f_2 + (\mu_1 - \mu_2) f_1 f_2 + \frac{c}{\lambda_1 \lambda_2} e^{-D_n} f_0 \bullet f_{12} = 0, \quad (\text{A10})$$

$$(D_y + \lambda_2^{-1} e^{-D_n} + \mu_2) f_1 \bullet f_{12} = 0, \quad (\text{A11})$$

$$(D_y + \lambda_1^{-1} e^{-D_n} + \mu_1) f_2 \bullet f_{12} = 0. \quad (\text{A12})$$

$$(D_z e^{-\frac{D_n}{2}} - \lambda_2 e^{\frac{D_n}{2}} + \gamma_2 e^{-\frac{D_n}{2}}) f_1 \bullet f_{12} = 0, \quad (\text{A13})$$

$$(D_z e^{-\frac{D_n}{2}} - \lambda_1 e^{\frac{D_n}{2}} + \gamma_1 e^{-\frac{D_n}{2}}) f_2 \bullet f_{12} = 0. \quad (\text{A14})$$

Thus, in order to prove Proposition 4, it suffices to show that

$$(\lambda_2^{-1} e^{-\frac{D_n}{2}+D_k} - 2D_z e^{\frac{D_n}{2}+D_k} - 2\gamma_2 e^{\frac{D_n}{2}+D_k} + \beta_2 e^{-\frac{D_n}{2}-D_k}) f_1 \bullet f_{12} = 0, \quad (\text{A15})$$

$$(\lambda_1^{-1} e^{-\frac{D_n}{2}+D_k} - 2D_z e^{\frac{D_n}{2}+D_k} - 2\gamma_1 e^{\frac{D_n}{2}+D_k} + \beta_1 e^{-\frac{D_n}{2}-D_k}) f_2 \bullet f_{12} = 0. \quad (\text{A16})$$

From

$$\frac{1}{\lambda_1} [(D_z e^{-\frac{D_n}{2}} - \lambda_1 e^{\frac{D_n}{2}} + \gamma_1 e^{-\frac{D_n}{2}}) f_0 \bullet f_1] f_2(n - \frac{1}{2}) \quad (\text{A17})$$

$$- \frac{1}{\lambda_2} [(D_z e^{-\frac{D_n}{2}} - \lambda_2 e^{\frac{D_n}{2}} + \gamma_2 e^{-\frac{D_n}{2}}) f_0 \bullet f_2] f_1(n - \frac{1}{2}) = 0, \quad (\text{A18})$$

we can deduce that [24]

$$[\lambda_2 D_z e^{(1/2)D_n} + \lambda_1 D_z e^{-(1/2)D_n} - 2\lambda_2 \gamma_1 e^{-(1/2)D_n} + 2\lambda_1 \gamma_2 e^{-(1/2)D_n}] f_1 \bullet f_2 - c D_z e^{-(1/2)D_n} f_0 \bullet f_{12} = 0 \quad (\text{A19})$$

Since f_1 and f_2 are two solutions of (35)-(36), we have, by using (46), (A10), (A15), (A19), and $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i, \beta_i)}$

f_i ($i = 1, 2$), that

$$\begin{aligned}
0 &= [(D_y e^{D_k} - 2D_z e^{D_n+D_k})f_1 \bullet f_1]e^{-D_k} f_2 \bullet f_2 - [(D_y e^{D_k} - 2D_z e^{D_n+D_k})f_2 \bullet f_2]e^{-D_k} f_1 \bullet f_1 \\
&= 2 \sinh(D_k)(D_y f_1 \bullet f_2) \bullet (f_1 f_2) - 2D_z \cosh\left(\frac{D_n}{2} + D_k\right)(e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad - 2 \sinh\left(\frac{D_n}{2} + D_k\right)[(D_z e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) - (e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet (D_z e^{-\frac{D_n}{2}} f_1 \bullet f_2)] \\
&= 2 \sinh(D_k)(D_y f_1 \bullet f_2) \bullet (f_1 f_2) + \frac{2}{\lambda_1} D_z \cosh\left(\frac{D_n}{2} + D_k\right)[(\lambda_2 e^{\frac{D_n}{2}} - \lambda_1 e^{-\frac{D_n}{2}})f_1 \bullet f_2] \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad - \frac{2}{\lambda_1} \sinh\left(\frac{D_n}{2} + D_k\right)[(\lambda_2 D_z e^{\frac{D_n}{2}} + \lambda_1 D_z e^{-\frac{D_n}{2}} f_1 \bullet f_2)] \bullet (e^{\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad \quad \quad + (D_z e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet [(\lambda_1 e^{-\frac{D_n}{2}} - \lambda_2 e^{\frac{D_n}{2}})f_1 \bullet f_2] \\
&= \frac{2c}{\lambda_1 \lambda_2} \sinh(D_k)(e^{-D_n} f_0 \bullet f_{12}) \bullet (f_1 f_2) + 2c D_z \cosh\left(\frac{D_n}{2} + D_k\right)(e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-\frac{D_n}{2}} f_0 \bullet f_{12}) \\
&\quad - \frac{4c\gamma_2}{\lambda_1} \sinh\left(\frac{D_n}{2} + D_k\right)(e^{-\frac{D_n}{2}} f_0 \bullet f_{12}) \bullet (e^{\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad - \frac{2c}{\lambda_1} \sinh\left(\frac{D_n}{2} + D_k\right)[(D_z e^{-\frac{D_n}{2}} f_0 \bullet f_{12}) \bullet (e^{\frac{D_n}{2}} f_1 \bullet f_2) + (e^{-\frac{D_n}{2}} f_0 \bullet f_{12}) \bullet (D_z e^{\frac{D_n}{2}} f_1 \bullet f_2)] \\
&= \frac{c}{\lambda_1} e^{-\frac{D_n}{2}} [\lambda_2^{-1}(e^{D_k - \frac{D_n}{2}} f_0 \bullet f_2) \bullet (e^{-D_k - \frac{D_n}{2}} f_1 \bullet f_{12}) - \lambda_2^{-1}(e^{-D_k - \frac{D_n}{2}} f_0 \bullet f_2) \bullet (e^{D_k - \frac{D_n}{2}} f_1 \bullet f_{12}) \\
&\quad + 2(e^{-D_k - \frac{D_n}{2}} f_0 \bullet f_2) \bullet (D_z e^{D_k + \frac{D_n}{2}} f_1 \bullet f_{12}) - 2(D_z e^{D_k + \frac{D_n}{2}} f_0 \bullet f_2) \bullet (e^{-D_k - \frac{D_n}{2}} f_1 \bullet f_{12}) \\
&\quad - 2\gamma_2(e^{D_k + \frac{D_n}{2}} f_0 \bullet f_2) \bullet (e^{-D_k - \frac{D_n}{2}} f_1 \bullet f_{12}) + 2\gamma_2(e^{-D_k - \frac{D_n}{2}} f_0 \bullet f_2) \bullet (e^{D_k + \frac{D_n}{2}} f_1 \bullet f_{12})] \\
&= \frac{c}{\lambda_1} e^{-\frac{D_n}{2}} [e^{-D_k - \frac{D_n}{2}} f_0 \bullet f_2] \bullet [(-\lambda_2^{-1} e^{-\frac{D_n}{2} + D_k} + 2D_z e^{\frac{D_n}{2} + D_k} + 2\gamma_2 e^{\frac{D_n}{2} + D_k} - \beta_2 e^{-\frac{D_n}{2} - D_k})f_1 \bullet f_{12}],
\end{aligned}$$

which implies that (A15) holds. Similarly we can prove (A16) also holds. Therefore we have completed the proof of the propositions 4.

A.3 Proof of Proposition 6:

we can show that

$$-D_z f_1 \bullet f_2 + (\gamma_2 - \gamma_1) f_1 f_2 - k f_0 f_{12} = 0, \quad (\text{A20})$$

$$(\mu_1 \lambda_2 e^{D_m} - \mu_2 \lambda_1 e^{-D_m}) f_1 \bullet f_2 = 2k e^{D_m + D_n} f_0 \bullet f_{12} \quad (\text{A21})$$

$$(D_z e^{\frac{D_n}{2}} - \lambda_2 e^{-\frac{D_n}{2}} - \gamma_2 e^{\frac{D_n}{2}}) f_1 \bullet f_{12} = 0, \quad (\text{A22})$$

$$(D_z e^{\frac{D_n}{2}} - \lambda_1 e^{-\frac{D_n}{2}} - \gamma_1 e^{\frac{D_n}{2}}) f_2 \bullet f_{12} = 0. \quad (\text{A23})$$

$$(\lambda_2 e^{D_m} + 2e^{D_m + D_n} + \mu_2 e^{-D_m}) f_1 \bullet f_{12} = 0, \quad (\text{A24})$$

$$(\lambda_2 e^{D_m} + 2e^{D_m + D_n} + \mu_2 e^{-D_m}) f_2 \bullet f_{12} = 0, \quad (\text{A25})$$

Thus, in order to prove Proposition 6, it suffices to show that

$$(D_z e^{-\frac{D_n}{2} - D_k - D_m} - \gamma_2 e^{-\frac{D_n}{2} - D_k - D_m} + \frac{1}{2\mu_2} e^{D_m + \frac{D_n}{2} - D_k} + \beta_2 e^{\frac{D_n}{2} + D_k + D_m}) f_1 \bullet f_{12} = 0, \quad (\text{A26})$$

$$(D_z e^{-\frac{D_n}{2} - D_k - D_m} - \gamma_1 e^{-\frac{D_n}{2} - D_k - D_m} + \frac{1}{2\mu_1} e^{D_m + \frac{D_n}{2} - D_k} + \beta_1 e^{\frac{D_n}{2} + D_k + D_m}) f_2 \bullet f_{12} = 0. \quad (\text{A27})$$

From

$$\frac{1}{\lambda_1} [(D_z e^{\frac{D_n}{2}} - \lambda_1 e^{-\frac{D_n}{2}} - \gamma_1 e^{\frac{D_n}{2}}) f_0 \bullet f_1] f_2(n + \frac{1}{2}) \quad (\text{A28})$$

$$- \frac{1}{\lambda_2} [(D_z e^{\frac{D_n}{2}} - \lambda_2 e^{-\frac{D_n}{2}} - \gamma_2 e^{\frac{D_n}{2}}) f_0 \bullet f_2] f_1(n + \frac{1}{2}) = 0, \quad (\text{A29})$$

we can deduce that [24]

$$k D_z e^{\frac{D_n}{2}} f_0 \bullet f_{12} + (\lambda_1 D_z e^{\frac{D_n}{2}} + \lambda_2 D_z e^{-\frac{D_n}{2}}) f_1 \bullet f_2 + (2\lambda_2 \gamma_1 e^{-\frac{D_n}{2}} - 2\lambda_1 \gamma_2 e^{\frac{D_n}{2}}) f_1 \bullet f_2 = 0. \quad (\text{A30})$$

Since f_1 and f_2 are two solutions of (57)-(59), we have, by using (62), (A21), (A30), (59) and $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i, \beta_i)}$ f_i ($i = 1, 2$), that

$$\begin{aligned}
0 &= \{[\sinh(D_m) \sinh(D_k) - 2D_z e^{D_n+D_m+D_k}] f_1 \bullet f_1\} (e^{-D_m-D_k} f_2 \bullet f_2) \\
&\quad - \{[\sinh(D_m) \sinh(D_k) - 2D_z e^{D_n+D_m+D_k}] f_2 \bullet f_2\} (e^{-D_m-D_k} f_1 \bullet f_1) \\
&= \sinh(D_k) (e^{D_m} f_1 \bullet f_2) \bullet (e^{-D_m} f_1 \bullet f_2) - 2D_z \cosh\left(\frac{D_n}{2} + D_m + D_k\right) (e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad - 2 \sinh\left(\frac{D_n}{2} + D_m + D_k\right) [(D_z e^{\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) + (D_z e^{-\frac{D_n}{2}} f_1 \bullet f_2) \bullet (e^{\frac{D_n}{2}} f_1 \bullet f_2)] \\
&= -\frac{1}{\mu_2 \lambda_1} \sinh(D_k) (e^{D_m} f_1 \bullet f_2) \bullet (\mu_1 \lambda_2 e^{D_m} - \mu_2 \lambda_1 e^{-D_m} f_1 \bullet f_2) \\
&\quad - \frac{2}{\lambda_1} D_z \cosh\left(\frac{D_n}{2} + D_m + D_k\right) [(\lambda_1 e^{\frac{D_n}{2}} - \lambda_2 e^{-\frac{D_n}{2}}) f_1 \bullet f_2] \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad - \frac{2}{\lambda_1} \sinh\left(\frac{D_n}{2} + D_m + D_k\right) \{[(\lambda_2 D_z e^{-\frac{D_n}{2}} + \lambda_1 D_z e^{\frac{D_n}{2}}) f_1 \bullet f_2] \bullet (e^{-\frac{D_n}{2}} f_1 \bullet f_2) \\
&\quad + (D_z e^{-\frac{D_n}{2}} f_1 \bullet f_2) \bullet [(\lambda_1 e^{\frac{D_n}{2}} - \lambda_2 e^{-\frac{D_n}{2}}) f_1 \bullet f_2]\} \\
&= \frac{k}{\lambda_1} e^{-\frac{D_n}{2}} \{-\mu_2^{-1} (e^{\frac{D_n}{2}+D_m+D_k} f_1 \bullet f_{12}) \bullet (e^{-D_k+D_m+\frac{D_n}{2}} f_0 \bullet f_2) + \mu_2^{-1} (e^{D_k+D_m+\frac{D_n}{2}} f_1 \bullet f_{12}) \bullet (e^{D_k+D_m+\frac{D_n}{2}} f_0 \bullet f_2) \\
&\quad - 2\gamma_2 (e^{-D_m-D_k-\frac{D_n}{2}} f_1 \bullet f_{12}) \bullet (e^{D_m+D_k+\frac{D_n}{2}} f_0 \bullet f_2) + 2\gamma_2 (e^{D_m+D_k+\frac{D_n}{2}} f_1 \bullet f_{12}) \bullet (e^{-\frac{D_n}{2}-D_m-D_k} f_0 \bullet f_2) \\
&\quad + 2(D_z e^{-\frac{D_n}{2}-D_m-D_k} f_1 \bullet f_{12}) \bullet (e^{\frac{D_n}{2}+D_k+D_m} f_0 \bullet f_2) - 2(e^{\frac{D_n}{2}+D_k+D_m} f_1 \bullet f_{12}) \bullet (D_z e^{-\frac{D_n}{2}-D_m-D_k} f_0 \bullet f_2)\} \\
&= \frac{k}{\lambda_1} e^{-\frac{D_n}{2}} \{[(\mu_2^{-1} e^{-D_k+D_m+\frac{D_n}{2}} - 2\gamma_2 e^{-\frac{D_n}{2}-D_m-D_k} + 2D_z e^{-\frac{D_n}{2}-D_m-D_k} + \beta_2 e^{\frac{D_n}{2}+D_m+D_k}) f_1 \bullet f_{12}] \\
&\quad \bullet (e^{\frac{D_n}{2}+D_m+D_k} f_0 \bullet f_2)\},
\end{aligned}$$

which implies that (A26) holds. Similarly we can prove (A27) also holds. Therefore we have completed the proof of the Propositions 6.

Appendix B. Hirota bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions a , b , c , and d .

$$D_y (D_z e^{D_n} a \bullet a) \bullet (e^{D_n} a \bullet a) = \sinh(D_n) (D_y D_z a \bullet a) \bullet a^2. \quad (\text{A31})$$

$$\sinh(D_m) (D_z e^{D_m+D_n} a \bullet a) \bullet (e^{D_m+D_n} a \bullet a) = \sinh(D_n + D_m) (D_z e^{D_m} a \bullet a) \bullet (e^{D_m} a \bullet a). \quad (\text{A32})$$

$$[D_x e^{\frac{1}{2} D_m} a \bullet a] [e^{\frac{1}{2} D_m} b \bullet b] - (e^{\frac{1}{2} D_m} a \bullet a) (D_x e^{\frac{1}{2} D_m} b \bullet b) = 2 \sinh\left(\frac{1}{2} D_m\right) (D_x a \bullet b) \bullet ab. \quad (\text{A33})$$

$$[e^{D_n+D_m} a \bullet a] [e^{D_m} b \bullet b] - [e^{D_n+D_m} b \bullet b] [e^{D_m} a \bullet a] = 2 \sinh\left(\frac{1}{2} D_n\right) (e^{\frac{1}{2} D_n+D_m} a \bullet b) \bullet (e^{-\frac{1}{2} D_n-D_m} a \bullet b). \quad (\text{A34})$$

$$\sinh\left(\frac{1}{2} D_m\right) a \bullet a = 0. \quad (\text{A35})$$

$$[e^{D_m-\frac{1}{2} D_n} a \bullet b] [e^{-\frac{1}{2} D_n-D_m} c \bullet a] = e^{D_m} (e^{-\frac{1}{2} D_n} a \bullet a) \bullet (e^{\frac{1}{2} D_n} b \bullet c) \quad (\text{A36})$$

$$\begin{aligned}
&[D_z e^{D_n+D_m} a \bullet a] [e^{D_m} b \bullet b] - [D_z e^{D_n+D_m} a \bullet a] [e^{D_m} b \bullet b] \\
&= 2D_z \cosh\left(\frac{1}{2} D_n\right) (e^{\frac{1}{2} D_n+D_m} a \bullet b) \bullet (e^{-\frac{1}{2} D_n-D_m} a \bullet b) \\
&\quad + [e^{D_n+D_m} a \bullet a] [D_z e^{D_m} b \bullet b] - [D_z e^{D_m} a \bullet a] [e^{D_m+D_n} b \bullet b]
\end{aligned} \quad (\text{A37})$$

$$\sinh(D_n + D_k) (D_y D_z a \bullet a) \bullet a^2 = D_y (D_z e^{D_n+D_k} a \bullet a) \bullet (e^{D_n+D_k} a \bullet a). \quad (\text{A38})$$

$$\begin{aligned}
& D_z \cosh\left(\frac{1}{2}D_n + D_k\right)(e^{\frac{1}{2}D_n}a \bullet b) \bullet (e^{-\frac{1}{2}D_n}a \bullet b) \\
&= \sinh\left(\frac{1}{2}D_n\right)[(D_z e^{\frac{1}{2}D_n + D_k}a \bullet b) \bullet (e^{-\frac{1}{2}D_n - D_k}a \bullet b) - (e^{\frac{1}{2}D_n + D_k}a \bullet b) \bullet (D_z e^{-\frac{1}{2}D_n - D_k}a \bullet b)]
\end{aligned} \tag{A39}$$

$$\begin{aligned}
& [D_z e^{D_n + D_k}a \bullet a][e^{-D_k}b \bullet b] - [D_z e^{D_n + D_k}b \bullet b][e^{-D_k}a \bullet a] \\
&= D_z \cosh\left(\frac{1}{2}D_n + D_k\right)(e^{\frac{1}{2}D_n}a \bullet b) \bullet (e^{-\frac{1}{2}D_n}a \bullet b) \\
&+ \sinh\left(\frac{1}{2}D_n + D_k\right)[(D_z e^{\frac{1}{2}D_n}a \bullet b) \bullet (e^{-\frac{1}{2}D_n}a \bullet b) - (e^{\frac{1}{2}D_n}a \bullet b) \bullet (D_z e^{-\frac{1}{2}D_n}a \bullet b)]
\end{aligned} \tag{A40}$$

$$\sinh(D_m + D_n + D_k)(D_z e^{D_m}a \bullet a) \bullet (e^{D_m}a \bullet a) = \sinh(D_m)(D_z e^{D_m + D_n + D_k}a \bullet a) \bullet (e^{D_m + D_n + D_k}a \bullet a). \tag{A41}$$

$$[e^{D_m - D_k}a \bullet a][e^{D_m + D_k}b \bullet b] - [e^{D_m - D_k}b \bullet b][e^{D_m + D_k}a \bullet a] = -2 \sinh(D_k)(e^{D_m}a \bullet b) \bullet (e^{-D_m}a \bullet b) \tag{A42}$$

$$\begin{aligned}
& [D_z e^{D_n + D_m + D_k}a \bullet a][e^{D_m + D_k}b \bullet b] - [D_z e^{D_n + D_m + D_k}b \bullet b][e^{D_m + D_k}a \bullet a] \\
&= 2 \sinh\left(\frac{1}{2}D_n + D_k + D_m\right)(D_z e^{\frac{1}{2}D_n}a \bullet b) \bullet (e^{-\frac{1}{2}D_n}a \bullet b) \\
&- 2 \sinh\left(\frac{1}{2}D_n\right)(e^{\frac{1}{2}D_n + D_k + D_m}a \bullet b) \bullet (D_z e^{-\frac{1}{2}D_n - D_k - D_m}a \bullet b)
\end{aligned} \tag{A43}$$

References

- [1] Leznov A N.: Graded Lie algebras, representation theory, integrable mappings and integrable systems, *Theor. Math. Phys.* **122** (2) 211-228 (2000)
- [2] M.Blaszak,K.Marciniak,*J.Math.Phys.***35** (1994)4661
- [3] X.B.Hu,Z.N.Zhu,*J.Math.Phys.***39** (1998) 4766
- [4] B.A.Kupershmidt,Discrete Lax equations and differential-difference calculus, *Asterisque* (123),212, (1985).
- [5] Levi D, Vinet L, and Winternitz P. (ed.): *Symmetries and Integrability of Difference Equations*, CRM Proceedings & Lecture Notes, Vol. 9, American Mathematical Society, (1996)
- [6] Clarkson P A and Nijhoff F W. (ed.): *Symmetries and Integrability of Difference Equations*, London Mathematical Society Lecture Note Series 255, Cambridge, New York, Cambridge University Press, (1999)
- [7] Levi D and Ragnisco O.(ed.): *SIDE III–Symmetries and Integrability of Difference Equations*, CRM Proceedings & Lecture Notes, Vol. 25, American Mathematical Society, (2000)
- [8] Hietarinta J, Nijhoff F W, and Satsuma J.(ed.): *Symmetries and integrability of difference equations*, Institute of Physics Publishing, Bristol, **34** 10337-10744 (2001)
- [9] Suris Y B.: *The problem of integrable discretization: Hamiltonian approach*. Progress in Mathematics, 219. Birkhäuser Verlag, Basel. (2003)
- [10] Hirota R.: Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation. *J. Phys. Soc. Japan.* **43** 1424-1433 (1977)
- [11] Hirota R.: Nonlinear partial difference equations. II. Discrete-time Toda equation. *J. Phys. Soc. Japan.* **43** 2074-2078 (1977)
- [12] Hirota R.: Nonlinear partial difference equations. III. Discrete sine-Gordon equation. *J. Phys. Soc. Japan* **43** 2079-2086 (1977)
- [13] Hirota R.: Nonlinear partial difference equations. IV. Bäcklund transformations for the discrete-time Toda equation. *J. Phys. Soc. Japan.* **45** 1321-1332 (1978)
- [14] Hirota R.: Discrete analogue of a generalized Toda equation. *J. Phys. Soc. Japan.* **50** 3785-3791 (1981)

- [15] Hirota, Ryogo; Iwao, Masataka.: Time-discretization of soliton equations. SIDE III—symmetries and integrability of difference equations (Sabaudia, 1998), 217–229, CRM Proc. Lecture Notes, 25, Amer. Math. Soc., Providence, RI, (2000)
- [16] Xing-Biao Hu and Guo-Fu Yu, Integrable discretizations of the (2+1)-dimensional sinh-Gordon equation. *J.Phys.A: Math.Theor.* **40**(42) 12645-12660 (2007)
- [17] Miura RM. (ed.): Bäcklund transformations, the inverse scattering method, solitons, and their applications. *Lecture Notes in Mathematics*, Vol. 515. Springer-Verlag, Berlin-New York. (1976)
- [18] Rogers C, Shadwick W F.: Bäcklund transformations and their applications. *Mathematics in Science and Engineering*, 161. Academic Press, Inc., New York-London. (1982)
- [19] Rogers C, Schief W K.: Bäcklund and Darboux transformations. *Geometry and modern applications in soliton theory*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge. (2002)
- [20] Hu X-B and Tam H-W.: Application of Hirota’s bilinear formalism to a two-dimensional lattice by Leznov, *Phys. Lett. A.* **276** 65-72 (2000)
- [21] Hirota R.: *Direct method in soliton theory* (in solitons, edited by R.K.Bullough and P.J.Caudrey, Springer, Berlin, 1980). (1980)
- [22] Hu X-B.: Nonlinear superposition formulae for the differential-difference analogue of the KdV equation and two-dimensional Toda equation. *J. Phys. A.* **27** 201–214 (1994)
- [23] Ma W-X, Hu X-B, Zhu S-M, Wu Y-T.: Bäcklund transformation and its superposition principle of a Blaszak-Marciniak four-field lattice. *J. Math. Phys.* **40** 6071-6086 (1999)
- [24] Hu X-B, Zhu Z-N.: Some new results on the Blaszak-Marciniak lattice: Bäcklund transformation and nonlinear superposition formula. *J. Math. Phys.* **39** 4766-4772 (1998)