Solving Two-Point Boundary Value Problems of Fractional Differential Equations by Spline Collocation Methods

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Abstract
We use collocation methods to solve fractional boundary value problems. Analytically, we study the existence and uniqueness theorem of the collocation solutions, and discuss some error estimates. We present numerical experiments to illustrate our results.

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1 Introduction
Fractional calculus and fractional differential equations (FDEs) are as old as the classical calculus (refer to [17] or [20] for a historical survey). They have been successfully applied to many fields, such as viscoelastic materials, signal processing, control, quantum mechanics, meteorology, finance, life sciences (see [4], [11], [16]-[21]). Many papers have focused on the analytical or the numerical study of fractional initial value problems (FIVPs) (see [5]-[7], [15], [17], [20]-[21], [29]-[32]). Comparatively, little attention has been paid to the fractional two-point boundary value problems (FBVPs). In this context, the existence of solutions of the Sturm-Liouville problem for an FDE, the Dirichlet-type FBVPs, a class of FBVPs with Riemann-Liouville fractional derivatives, some kind of FBVPs with Caputo’s derivatives, and a coupled system of nonlinear FDEs have been considered by Aleroev [1] and Nakhushev [18], Kilbas and Trujillo [13], Zhang [33], Bai and Lü [2], and Su [28] respectively; the least squares finite-element technique is employed by Roop and his coworkers to solve some kind of FBVPs [9], [10]; and the Adomian decomposition method is used by Jafari and Daftardar-Gejji to find approximate and positive solutions of a kind of FBVPs with Caputo’s fractional derivative [12]. In [14], [19], we have studied the existence and uniqueness of solutions of some kinds of FBVPs with Caputo’s

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derivatives and Riemann-Liouville derivatives, respectively. Moreover, we have designed shooting methods to simulate these models successfully. Besides these cited works, few more contributions have been made to the analytical and numerical study of the solutions of FBVPs.

In this paper, we use cubic spline collocation methods to solve the following linear FBVPs

\[
L[y] = y''(t) + \sum_{i=1}^{m} p_i(t) \cdot R_L^{\gamma_i}y(t) + p_{m+1}(t)y(t) = f(t), \quad a < t < b, \quad 0 < \gamma_i \leq 1, \quad (1.1)
\]

\[
y(a) = \alpha, \quad y(b) = \beta, \quad (1.2)
\]

where \( p_i(t) \in C[a,b] \).

Early contributions to collocation with polynomial functions for BVPs were due to Russian authors (see [25], [26], [31]). They considered expressions of the kind

\[
L[y] = y^{(m)}(t) + \sum_{i=0}^{m-1} q_i(t)y^{(i)}(t) = f(t), \quad a < t < b, \quad (1.3)
\]

\[
\sum_{j=0}^{m-1} [\alpha_{ij}y^{(j)}(a) + \beta_{ij}y^{(j)}(b)] = \eta_i, \quad (\alpha_{ij}, \beta_{ij}, \eta_i = const; 1 \leq i \leq m) \quad (1.4)
\]

where \( q_i(t) \in C[a,b] \). Later on, Russell and Shampine [24] developed collocation with piecewise polynomial functions to solve BVPs and such a spline collocation method is still today a basic and important numerical tool.

Based on the successful application of spline collocation to BVPs such as (1.3-1.4), our aim is to develop a cubic spline collocation method to solve FBVPs given by (1.1-1.2). Spline collocation methods have been used to solve some FIVPs (see [3], [23]) but, to our best knowledge, they have not been used for FBVPs. We will see in what follows that the cubic spline collocation is also an efficient numerical method for the FBVPs (1.1-1.2).

This paper is organized as follows: first, in Section 2, we deal with some necessary preliminaries, we give the integrated form of the FBVPs (1.1-1.2) and show the equivalence of the two forms. In Section 3 we design the collocation method to solve the FBVPs (1.1-1.2) and give the corresponding error analysis. Finally, in section 4 we present the results of numerical experiments.

2 Preliminaries

2.1 Basic Definitions

We start introducing the definitions of the Riemann-Liouville fractional derivative and fractional integral:

**Definition 2.1** (see [32]) Let be \( \gamma > 0 \) and \( n \in \mathbb{N} \) such that \( n - 1 < \gamma \leq n \), and \( t \in [a,b] \). We define the Riemann-Liouville differential operator of order \( \gamma \), \( R_L^{\gamma} \), acting on \( y(t) \in AC^n[a,b] \), as:

\[
R_L^{\gamma}y(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-\tau)^{n-\gamma-1}y(\tau) \, d\tau, \quad (2.1)
\]

where \( AC^n[a,b] \) is the set of functions \( y \) with continuous derivatives up to order \( n - 1 \) on \( [a,b] \) and such that \( y^{(n-1)}(t) \) is absolutely continuous in \( [a,b] \).
Definition 2.2 (see [32]) Let be \( \gamma > 0 \). We define the Riemann-Liouville fractional integral operator of order \( \gamma \), \( J_{a}^{\gamma} \), acting on \( y(t) \in L_{1}[a,b] \), as:
\[
J_{a}^{\gamma}y(t) = \frac{1}{\Gamma(\gamma)} \int_{a}^{t} (t-\tau)^{\gamma-1} y(\tau) d\tau ,
\]
where \( L_{1}[a,b] \) is the set of measurable functions in \([a,b]\) with \( \int_{a}^{b} |y(t)| dt < \infty \).

We also define Caputo’s fractional derivatives:

Definition 2.3 (see [32]) Let be \( \gamma > 0 \) and \( n \in \mathbb{N} \) such that \( n - 1 < \gamma \leq n \), and \( t \in [a,b] \). We define the Caputo differential operator of order \( \gamma \), \( C_{a}^{\gamma}D_{t}^{\gamma} \), acting on \( y(t) \in C^n[a,b] \), as:
\[
C_{a}^{\gamma}D_{t}^{\gamma}y(t) = J_{a}^{n-\gamma}y^{(n)}(t)
\]
where \( C^n[a,b] \) is the set of continuously differentiable functions up to order \( n \).

An important property relates \( J_{a}^{\gamma} \) and \( R_{a}^{\gamma}D_{t}^{\gamma} \):

Theorem 2.4 (see [32]) Let be \( \gamma > 0 \). If \( f \) is continuous, then
\[
R_{a}^{\gamma}D_{t}^{\gamma}J_{a}^{\gamma}f = f .
\]

2.2 The Equivalent Integrated Form of FBVPs (1.1-1.2)

We study now an equivalent integrated form for our FBVPs (1.1-1.2), that will be suitable to establish the error estimates of the spline collocation methods.

Without loss of generality, we only need to consider the case of homogeneous boundary conditions:
\[
y(a) = 0, \quad y(b) = 0 ,
\]
for suppose we have a solution \( y(t) \) of (1.1) subject to the nonhomogeneous boundary condition (1.2): defining \( u(t) = y(t) - q(t) \), with \( q(t) \) a function in \( C^2[a,b] \) such that \( q(a) = \alpha, q(b) = \beta \), the problem is transformed into solving \( Lu = f(t) - Lq(t) = \tilde{f}(t) \) subject to the homogeneous boundary conditions \( u(a) = u(b) = 0 \).

On the other hand the following problem
\[
C_{a}^{\gamma}D_{t}^{\gamma}y(t) + py(t) = g(t) , \quad a < t < b , \quad 1 < \gamma \leq 2 ,
\]
\[
y(a) = 0 , \quad y(b) = 0 ,
\]
where \( p \) is a constant, can be transformed to the previous form (1.1-1.2), provided that \( y(t) \in C^2[a,b] \) and \( g(t) \in AC[a,b] \). According to Definition 2.3, (2.7) is equivalent to
\[
J_{a}^{2-\gamma}y''(t) + py(t) = g(t).
\]

Acting with \( R_{a}^{\gamma}D_{t}^{2-\gamma} \) on both sides of this equation and using Theorem 2.4, we finally obtain
\[
y''(t) + p \cdot R_{a}^{\gamma}D_{t}^{2-\gamma}y(t) = \tilde{g}(t) , \quad a < t < b , \quad 0 < 2 - \gamma \leq 1
\]
\[
y(a) = 0 , \quad y(b) = 0 ,
\]
with \( \tilde{g}(t) = R_{a}^{\gamma}D_{t}^{2-\gamma}g(t) \).
In [18], Nakhushhev has proven the existence and uniqueness for the solutions of FBVPs (1.1-1.2). Let be \( y''(t) = v(t) \), then the solution of (1.1-1.2) is equivalent to the solution of the BVPs

\[
  y''(t) = v(t), \quad y(a) = 0, \quad y(b) = 0. \tag{2.9}
\]

Via the Green’s function, the solution of (2.9-2.10) is uniquely specified as

\[
y(t) = \int_a^b G(t, s)v(s) \, ds = G \, v(t), \tag{2.11}
\]

where

\[
  G(t, s) = \begin{cases} 
    (t - s) - \frac{(t-a)(b-s)}{(b-a)}, & a \leq s \leq t \leq b, \\
    -\frac{(t-a)(b-s)}{(b-a)}, & a \leq t \leq s \leq b
  \end{cases} \tag{2.12}
\]

is the Green’s function and \( G \) is a compact operator, since \( G(t, s) \) is continuous in \([a, b] \times [a, b]\).

We can easily obtain the following lemma:

**Lemma 2.5**

\[
  ^{RL}_a D_t^\gamma y(t) = ^{RL}_a D_t^\gamma \int_a^b G(t, s)v(s) \, ds = \int_a^b (^{RL}_a D_t^\gamma)G(t, s)v(s) \, ds = ^{RL}_a D_t^\gamma Gv(t), \quad 0 < \gamma \leq 1.
\]

**Proof:** Let be \( \int_a^b G(t, s)v(s) \, ds = I(t) \), then

\[
  ^{RL}_a D_t^\gamma y(t) = ^{RL}_a D_t^\gamma \int_a^b G(t, s)v(s) \, ds
  = \frac{d}{dt} \int_a^t \frac{1}{\Gamma(1-\gamma)} (t-\tau)^{-\gamma} I(\tau) \, d\tau
  = \frac{d}{dt} \int_a^t \frac{1}{\Gamma(2-\gamma)} I(\tau) \, d(t-\tau)^{1-\gamma}
  = \frac{d}{dt} \left[ \frac{1}{\Gamma(2-\gamma)} I(\tau) \right]_a^t + \frac{1}{\Gamma(2-\gamma)} \int_a^t (t-\tau)^{1-\gamma} I'(\tau) \, d\tau
  = \frac{d}{dt} \left[ \frac{1}{\Gamma(2-\gamma)} \int_a^t (t-\tau)^{1-\gamma} I'(\tau) \, d\tau \right]
  = \frac{d}{dt} \left[ \frac{1}{\Gamma(2-\gamma)} \int_a^b (t-\tau)^{1-\gamma} \frac{\partial}{\partial \tau} G(\tau, s)v(s) \, d\tau \right]
  = \frac{d}{dt} \left[ \int_a^b \left( \frac{1}{\Gamma(2-\gamma)} \int_a^t (t-\tau)^{1-\gamma} \frac{\partial}{\partial \tau} G(\tau, s) \, d\tau \right) v(s) \right]
  = \int_a^b \frac{d}{dt} \left( \frac{1}{\Gamma(2-\gamma)} \int_a^t (t-\tau)^{1-\gamma} dG(\tau, s) \right) v(s) \, ds
  = \int_a^b \frac{d}{dt} \left( \frac{1}{\Gamma(2-\gamma)} (t-\tau)^{1-\gamma} G(\tau, s) \right) \bigg|_a^t - \frac{1}{\Gamma(2-\gamma)} \int_a^t G(\tau, s)(t-\tau)^{1-\gamma} \, ds
  = \int_a^b \frac{d}{dt} \left( \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-\tau)^{-\gamma} G(\tau, s) \, d\tau \right) v(s) \, ds
  = \int_a^b \left( ^{RL}_a D_t^\gamma \right) G(t, s)v(s) \, ds.
\]

Due to this lemma, we also have the following corollary:

**Corollary 2.6**

The operator \( ^{RL}_a D_t^\gamma G \) is compact.
Substituting (2.11) into (1.1), we get

\[ v(t) + \sum_{i=1}^{m} p_i(t) \cdot \frac{d^2}{dt^2} G(t) + p_{m+1}(t) \cdot Gv(t) = f(t). \]

We can express this as

\[ v(t) + \int_{a}^{b} K(t, s) v(s) \, ds = f(t), \quad \text{(2.13)} \]

with the kernel function

\[ K(t, s) = \sum_{i=1}^{m} p_i(t) \cdot \frac{d^2}{dt^2} G(t, s) + p_{m+1}(t) G(t, s). \]

According to Corollary 2.6, it is also a compact operator. Then solving FBVPs (1.1-1.2) is equivalent to solving BVPs (2.9-2.10) and the integral equation (2.13). For convenience, we rewrite (2.13) as

\[ (I + K)v = f. \quad \text{(2.14)} \]

### 3 Spline Collocation Methods

In this section, we will use a collocation method to solve the FBVPs (1.1-1.2), discuss the existence and uniqueness of the collocation solution and its error analysis.

Let be \( \pi_n : a = t_0 < t_1 < \cdots < t_n = b \) a partition of \([a, b]\). \( L(\pi_n, 3, 2) \) denote the space of cubic splines that satisfy the boundary conditions (1.2). That is, if \( y_n(t) \in L(\pi_n, 3, 2) \), then \( y_n(t) \) is a polynomial of degree 3 on each subinterval of \( \pi_n \) such that \( y_n(a) = y_n(b) = 0 \). For any \( y_n(t) \in L(\pi_n, 3, 2) \), \( y_n''(t) \in C[a, b] \) is a piecewise Lagrange interpolation polynomial. We solve the FBVPs (1.1-1.2) with collocation method

\[ Ly_n(t_i) = f(t_i), \quad 0 \leq i \leq n, \quad \text{(3.1)} \]

\[ y_n(a) = 0, \quad y_n(b) = 0, \quad \text{(3.2)} \]

where \( y_n(t) \in L(\pi_n, 3, 2) \) is the collocation solution we seek. Similarly to the analysis done above for the continuous case, it is easy to show that using collocation method to solve problem (3.1-3.2) is equivalent to applying a collocation method to the integral equations

\[ v_n(t_i) + Kv_n(t_i) = f(t_i), \quad 0 \leq i \leq n \quad \text{(3.3)} \]

where \( v_n(t) \) is a piecewise Lagrange polynomial on \( \pi \), and then finding \( y_n(t) \in L(\pi_n, 3, 2) \) that satisfies

\[ y_n''(t) = v_n(t), \]

\[ y_n(a) = 0, \quad y_n(b) = 0. \]

Define a linear projection \( P_n \) which maps each continuous function into its piecewise Lagrange interpolating polynomial. Let \( P_n f \) be a piecewise Lagrange polynomial of \( f \), then the necessary and sufficient condition for equations (3.3) to have \( v_n(t) \) as a solution is

\[ P_n v_n + P_n Kv_n = P_n f. \quad \text{(3.4)} \]

Since

\[ v_n = P_n v_n, \]
bounded, we have \( \| \| \leq \| \) \( N \)

Proof: Let \( B \).

Theorem 3 proves the lemma.

\[ C \]

Lemma 3 holds.

\[ V \]

Studying the existence and uniqueness of the solution for (3.3) is, thus, equivalent to consider that equations (3.5) have a unique solution in \( C[a,b] \) when the partition \( \pi \) is small enough. In the following, we will prove the existence and uniqueness of the solution of equations (3.5).

Firstly, let us present some preparatory results. We will use, in what follows, the norm for \( y(t) \in C[a,b] \) given by

\[ \| y \| = \max_{a \leq t \leq b} |y(t)|. \] (3.6)

**Lemma 3.1** (see [22]) The operators \( P_n \) converge strongly to the identity operator \( I \), and are uniformly bounded in \( C[a,b] \).

**Lemma 3.2** If \( n \to \infty \), then \( \| K - P_n K \| \to 0 \).

Proof: Let \( B \) be a unit ball in \( C[a,b] \), \( B = \{ v \in C[a,b] : \| v \| \leq 1 \} \). Since \( K \) is a compact operator, \( K[B] \) is bounded. That is, for given \( \varepsilon > 0 \), \( \exists N_\varepsilon = \{ y_1, y_2, \ldots, y_q \} \subset C[a,b] \), such that for every given \( v \in B \), \( y_i \in N_\varepsilon \) exists, and satisfies \( \| K v - y_i \| < \varepsilon \). Then

\[ \| K v - P_n K v \| = \| (I - P_n) K v \| \leq \| (I - P_n)(K v - y_i) \| + \| (I - P_n)y_i \| \leq \| (I - P_n) \| \varepsilon + \| (I - P_n)y_i \|. \]

Let \( N_0 \) be large enough, for all \( y_i \in N_\varepsilon \) and \( n \geq N_0 \), we have \( \| (I - P_n)y_i \| < \varepsilon \). Since \( P_n \) is uniformly bounded, we have \( \| (I - P_n) \| \leq 1 + \mu \). Thus, for \( n \geq N_0 \), we have \( \| K v - P_n K v \| \leq (2 + \mu)\varepsilon \), which proves the lemma.

**Theorem 3.3 (Neumann Theorem)** Let \( [V] \) denote the set of linear bounded operators mapping \( V \) to \( V \). Assume that \( S, T \in [V] \) and \( T^{-1} \in [V] \). If \( \Delta = \| T^{-1} \| \| S - T \| < 1 \), then \( S^{-1} \) exists such that \( S^{-1} \in [V] \), and the bound

\[ \| S^{-1} - T^{-1} \| \leq \frac{\| T^{-1} \| ^2 \| T - S \|}{1 - \Delta} \]

holds.

With the previous lemmas and theorem, we can prove the following result which supposes that equations (3.5) have a unique solution.

**Theorem 3.4** If \( N_0 \) is large enough, then \( \{ (I + P_n K)^{-1} : n \geq N_0 \} \) exists and consists of a sequence of bounded linear operators. In other words, for a constant \( \delta \) independent of \( N_0 \) and \( y \in C[a,b] \), if \( n \geq N_0 \), then

\[ \| (I + P_n K)^{-1} y \| \leq \delta \| y \| \]

holds.

**Theorem 3.5** Let \( y_n(t) \in Y_n \) be the solution of (3.1-3.2) and \( y(t) \) that of the FBVPs (1.1-1.2). If \( n \geq N_0 \), then we have

\[ \| y - y_n \| \leq A_k \| y^{(k+2)} \| h^k \], for \( y \in C^{k+2}[a,b] \), \( 1 \leq k \leq 2 \), \hspace{1cm} (3.7)

\[ \| y - y_n \| \leq A_0 \omega(y^{''}, h) \], for \( y \in C^{2}[a,b] \), \hspace{1cm} (3.8)
where $A_k, A_0$ are constants and independent of $y$ and $h$, and

$$\omega(\phi, h) = \sup\{|\phi(t + \hat{h}) - \phi(t)| : t, t + \hat{h} \in [a, b], |\hat{h}| \leq h\}.$$  \hspace{0.5cm} (3.9)

Proof: Let $v(t) = y''(t)$, then $v(t)$ is the solution of

$$v + Kv = f,$$  \hspace{0.5cm} (3.10)

where $y(t)$ is the solution of the FBVPs (1.1-1.2). Using the projection operator $P_n$ on both sides of (3.10) gives

$$P_n v + P_n K v = P_n f.$$  \hspace{0.5cm} (3.11)

We further add $v$ to both sides of (3.11) and then move $P_n v$ to the right-hand side and get

$$v + P_n K v = P_n f + v - P_n v.$$  \hspace{0.5cm} (3.12)

Subtracting now (3.12) from (3.5) and acting with operator $(I + P_n K)^{-1}$ on both sides yields finally

$$v - v_n = (I + P_n K)^{-1} (v - P_n v).$$  \hspace{0.5cm} (3.13)

Noticing that $y'' = v$, $y = Gv$, and $y_n = Gv_n$, and using operator $G$ act on both sides of (3.13) gives

$$y - y_n = G (I + P_n K)^{-1} (y'' - P_n y'').$$  \hspace{0.5cm} (3.14)

Since operators $G$ and $(I + P_n K)^{-1}$ are bounded we have

$$\|y - y_n\| \leq \|G\| \cdot \|(I + P_n K)^{-1}\| \cdot \|y'' - P_n y''\|.$$  \hspace{0.5cm} (3.15)

According to the classic theory of interpolation (see [22]), we have

$$\|y'' - P_n y''\| \leq \eta_k \|y^{(k+2)}\| h^k, \text{ for } y \in C^{k+2}[a,b], \ 1 \leq k \leq 2$$
$$\|y'' - P_n y''\| \leq \eta_0\omega(y'', h), \text{ for } y \in C^2[a,b],$$

Using Theorem 3.4, we have $\|(I + P_n K)^{-1}\| \leq \delta$ for $n \geq N_0$. We finish our proof setting $A_0 = \delta\eta_0\|G\|$ and $A_k = \delta\eta_k\|G\|, \ k = 1,2$.

4 Numerical experiments

In this section, we give two numerical examples to show the feasibility and validity of cubic spline collocation methods for the FBVPs (1.1-1.2).

Example 1  Consider the following linear FBVP

$$y''(t) + \sin(t) \int_0^1 RL D_0^{0.5} y(t) + ty(t) = f(t), \ 0 < t < 1,$$
$$y(0) = y(1) = 0,$$

where

$$f(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin(t) \left( \frac{32768}{6435\sqrt{\pi}} t^{7.5} - \frac{2048}{429\sqrt{\pi}} t^{6.5} \right).$$

One can easily check that $y(t) = t^8 - t^7$ is the unique analytical solution. The rates of convergence and maximum errors between the numerical solution (obtained by using spline collocation method
Example 2 Consider the following nonlinear FBVP

\[ C^{1.5}_0 D_t^5 y(t) + y(t) = t^5 - t^4 + \frac{128}{7\sqrt{\pi}} t^{3.5} - \frac{64}{\sqrt{\pi}} t^{2.5}, \quad 0 < t < 1, \]
\[ y(0) = y(1) = 0. \]

It can be transformed into

\[ y''(t) + \frac{R}{63\sqrt{\pi}} D_t^{0.5} y(t) = \frac{256}{63\sqrt{\pi}} t^{4.5} - \frac{128}{35\sqrt{\pi}} t^{3.5} + 20t^3 - 12t^2, \quad 0 < t < 1, \]
\[ y(0) = y(1) = 0. \]

One can easily check that \( y(t) = t^4(t - 1) \) is the analytical solution. The rates of convergence and maximum errors between the numerical solution (obtained by using spline collocation method mentioned above) and the analytical solution are given in table 2.

| stepsize \( h = 1/n \) | \( \max_{1 \leq i \leq n} |y(t_i) - y_n(t_i)| \) | rates of convergence |
|-------------------------|---------------------------------|---------------------|
| 1/8                     | 1.5417e-002                     | 2.0307              |
| 1/16                    | 3.7731e-003                     | 2.0307              |
| 1/32                    | 9.3823e-004                     | 2.0077              |
| 1/64                    | 2.3444e-004                     | 2.0007              |
| 1/128                   | 5.8595e-005                     | 2.0004              |

| stepsize \( h = 1/n \) | \( \max_{1 \leq i \leq n} |y(t_i) - y_n(t_i)| \) | rates of convergence |
|-------------------------|---------------------------------|---------------------|
| 1/8                     | 6.9299e-003                     | 1.9962              |
| 1/16                    | 1.7368e-003                     | 1.9962              |
| 1/32                    | 4.3646e-004                     | 1.9925              |
| 1/64                    | 1.0914e-004                     | 1.9997              |
| 1/128                   | 2.7287e-005                     | 1.9999              |

The numerical results given in Tables 1 and 2 show that the cubic spline collocation method has a 2-order rate of convergence for example 1 and example 2, which supports our theorem 3.5, since \( y(t) \in C^4[0,1] \). It is, thus, a successful tool to solve the linear fractional boundary value problems (1.1-1.2) and (2.7-2.8).

5 Conclusion

In this paper, we have designed cubic spline collocation method to solve two classes of special fractional boundary value problems, i.e., (1.1-1.2) and (2.7-2.8). At the same time, we have proved its validity by theoretical analysis and numerical experiments.

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