Hierarchical control based output synchronization of coexisting attractor networks *

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Abstract:

This paper introduces the concept of hierarchical control based output synchronization of coexisting attractor networks. Under the new framework, each dynamical node is made passive through intra control at first, which is only related with node dynamics itself. Then each dynamic node is looked as one agent, and on account of that, the solution of output synchronization of coexisting attractor networks is transformed into multi-agent consensus problem, which can be made possible in helps of local interaction among each agent’s neighbors, this distributed working ways of coordination is named as inter control action, and it is contributed by its nearest neighbors, and can be only determined by the topology structure of the network. As long as the network is connected and balanced, network output synchronization will come true via synergy of intra control and inter control actions, which can be proved theoretically through building of a composite Lyapunov function from its individual node ones. In case of completeness, we give several illustrative examples, where the coexisting network consists with 5 identical Newton-Leipnik nodes, their topology structure varies from star connection, ring connection to chain connection, which are provided to further illustrate the novelty and the efficacy of the propose scheme. For its independence of linearization around any equilibrium, all the results are global, this is strikingly attractive. And even further, how to cooperatively manipulate node dynamics and networks topology in network system, which is the fundamental question in network control, is partially answered in some extent in this paper.

Keywords:
Hierarchical control, passive control, composite Lyapunov function, the Newton-Leipnik equation attractor

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1. Introduction

Various large-scale and complicated systems can be modeled by networks, including the Internet, WWW, genetic networks, social network, and many others. The most striking recent advances are the occurrence of the small world network model\cite{1} and scale free network model\cite{2}, which have been declared to be closer to most real-world networks as compared with the earlier random-graph model\cite{3}. Since that time, many scientists and scholars contributed to this topic, and answered some questions like network stability along with stabilization\cite{4-7}, network spreading\cite{8} and network growing\cite{9}. And besides that, network synchronization was also concentrated, for its wide spread applications in various areas of physical and biological sciences\cite{10-20}. Among the already available results, synchronization with topology of the networks, synchronization cost, local synchronization with and without time varying topology for both continuous and discrete networks, phase synchronization and adaptive synchronization were investigated in detail. It is undoubtedly that those contributions are important for the further research. However, less attention was paid for the physical characteristics of the network nodes hitherto, and that may be one of the possible reasons for the complexity of the results.

In this paper, node physical dynamics as well as topology of the networks is concerned equivalently, synchronization is realized by the cooperative manipulation of the node physical characteristics and the network topology. The added control action consists of two different parts, one is only associated with the node dynamics itself, which is named as intra control action, and the other one is related with the node neighbor’s outputs, which is named as inter control action. While the intra control action makes the single node dynamics passive, the inter control action drives the whole network output synchronization. Unlike the already existing network output synchronization results, our method don’t use local linearization among equilibrium, this makes our results stand globally, and even more important contribution of our method is that, we turn network synchronization problem into multi-agent consensus one, and in this way, together with in advance assumptions of topological characteristics like graph connectivity and balance, coexisting attractor networks output synchronization comes naturally. In case of theoretical rigidity and completeness, the composite Lyapunov function, which is the convex composition of the individual node counterpart, is employed to prove the rightness of the suggested strategy.

The rest of the paper is organized as follows. In Section 2, some preliminary definitions and lemmas of being made passive are presented, which aims at elucidating the intra control action of the hierarchical control. In Section 3, the inter control action of the hierarchical control is discussed, which pays much attention to its distributive working mechanisms. Examples will be provided in Section 4 to further illustrate the efficiency of the hierarchical policy. Concluding remarks are given in Section 5.

2. Preliminaries

Let us consider a nonlinear affine system
\[
\dot{x} = f(x) + g(x)u, \\
y = h(x),
\]
where the state \( x \in \mathbb{R}^n \), the input \( u \in \mathbb{R}^m \), and output \( y \in \mathbb{R}^m \). \( f \) and together with the \( m \) column of \( g \) are smooth vector fields, \( h \) is a smooth mapping. We suppose that the vector field \( f \) has at least one equilibrium point, without loss of generality, we can assume that the equilibrium point is \( x = 0 \). If the equilibrium is not at \( x = 0 \), through coordinate shift we can transfer the equilibrium points to \( x = 0 \).

**Definition 1** The system (1) is said to be passive if there exists a real constant \( \beta \) such that for \( \forall t \geq 0 \), one of the inequalities among (2a)-(2c) can hold

\[
\int_0^t u^T(\tau)y(\tau)d\tau \geq \beta \quad (2a)
\]

\[
\int_0^t u^T(\tau)y(\tau)d\tau + \beta \geq \int_0^t y^T(\tau)y(\tau)d\tau, \; \rho > 0.
\]

The passive system (1) with storage function \( V(x) \) can be further said to be strictly passive if there exists a positive definite function \( S(x) \), such that for \( \forall x \in X, \forall t \geq 0 \), (2d) stands

\[
V(x) - V(x_0) \leq \int_0^t y^T(\tau)u(\tau)d\tau.
\]

\[
V(x) > 0, \forall x \neq \{0\}, V(0) = 0.
\]

The physical meaning of being passive is that the energy of the nonlinear system can be increased only through the supply from the external source. The stability properties of passive system are well known \([21-23]\). Once the system has been rendered passive, the controller law such as \( u = -\phi(y) \) can asymptotically stabilize the equilibrium of origin \( x = 0 \) of the nonlinear system (1), where \( \phi(y) \) needs to satisfy \( y^T\phi(y) > 0 \). \( \phi(y) \) can be selected in linear form.

In this paper we will use hierarchical control scheme, which integrates the passive technique and coordination strategy to control a coexisting chaotic system networks: the Newton-Leipnik chaotic system networks, all nodes are identical, i.e.
each node of the networks has the same dynamics, which is obtained by Newton R B and Leipnik T A through modifying Euler’s body equations with the addition of a linear feedback in 1981[24], its control problem has been investigated by several authors[25-27].

If the system (1) has relative degree \([1, \ldots, 1]\) at \(x = 0\) (i.e. \(L_y h(0)\) is nonsingular) and the distribution spanned by the vector field \(g_i(x), \ldots, g_n(x)\) is involutive, then it can be represented as the so-called normal form

\[
\begin{align*}
\dot{z} &= f_0(z) + p(z, y)y, \\
\dot{y} &= h(z, y) + a(z, y)u. 
\end{align*}
\]

(3)

The nonlinear system (3) may be rendered passive by a state feedback of the form \([4]\)

\[u = \alpha(x) + \beta(x)v.\]

(4)

Now we use this idea to make the Newton-Leipnik system (5) passive.

\[
\begin{align*}
\dot{x}_1 &= -\gamma x_1 + x_2 + 10x_2 x_3, \\
\dot{x}_2 &= -x_1 - 0.4 x_2 + 5 x_3 x_5, \\
\dot{x}_3 &= \sigma x_3 - 5 x_3 x_2. 
\end{align*}
\]

(5)

(5) has two coexisting chaotic attractors when \(\gamma = 0.4\) and \(\sigma = 0.175\), the equilibrium points of the system (5) with the nominal parameters are

\[
\begin{align*}
x_1^{(1)} &= (0, 0, 0)^T, \\
x_1^{(2)} &= (0.2390, 0.0308, 0.2103)^T, \\
x_1^{(3)} &= (-0.2390, -0.0308, 0.2103)^T, \\
x_1^{(4)} &= (0.0315, -0.1224, -0.1103)^T, \\
x_1^{(5)} &= (-0.0315, 0.1224, -0.1103)^T. 
\end{align*}
\]

All these equilibrium points are unstable. Note that the points \(x_1^{(2)}\) and \(x_1^{(3)}\) lie in the eye-centers of UA, while \(x_1^{(4)}\) and \(x_1^{(5)}\) are situated in the eye-centers of LA, and its phase locus is as Fig.1.

| Fig.1 The double strange attractors of the Newton-Leipnik system |

The Newton-Leipnik equation (5) is already in the normal form of (3), where
Choose a storage function candidate

\[ V(z, y) = W(z) + \frac{1}{2} y^2. \]  

(7)

Where \( W(z) = \frac{1}{2} (z_1^2 + z_2^2) \) is Lyapunov function with \( W(0) = 0 \). The zero dynamics of the Newton-Leipnik system (6) describe those internal dynamics, which are consistent with external constraint \( y = 0 \), i.e.,

\[ \dot{z} = f_0(z). \]  

(8)

Consider (8) and because \( \gamma \) and \( \sigma \) are positive constants

\[ \frac{d}{dt} W(z) = [z_1, z_2][-\gamma z_1 + z_2, -z_1 - 0.4 z_2]^T, \]

\[ = -\gamma z_1^2 - 0.4 z_2^2 \leq 0. \]  

(9)

The zero dynamics of the Newton-Leipnik system is Lyapunov stable, i.e., the Newton-Leipnik system is minimum phase. The derivative of \( V(z, y) \) along the trajectory of the Newton-Leipnik system (6) is

\[ \frac{d}{dt} V(z, y) \leq \frac{\partial}{\partial z} W(z) \dot{z} + y \dot{y}, \]

\[ = \frac{\partial}{\partial z} W(z) f_0(z) + \frac{\partial}{\partial z} W(z) p(z, y) y + [b(z, y) + a(z, y) u] y. \]  

(10)

As the Newton-Leipnik system is minimum phase

\[ \frac{\partial}{\partial z} W(z) f_0(z) \leq 0 \]

Then (10) becomes

\[ \frac{d}{dt} V(z, y) \leq \frac{\partial}{\partial z} W(z) p(z, y) y + [b(z, y) + a(z, y) u] y. \]
If we select the feedback control (4) as (11) and consider (6):

$$u = a^{-1}(z, y) [-h^T(z, y) - \frac{\partial W}{\partial z} p(z, y) - \lambda y + v],$$

$$= -\sigma y - 10z_1z_2 - \lambda y + v. \quad (11)$$

where $\lambda$ is a positive constant and $v = -\varphi(y)$ s.t. $y\varphi(y) > 0$. The above inequality can be rewritten as

$$\frac{d}{dt} V(z, y) \leq vy - \lambda y^2. \quad (12)$$

Then, taking integration over both sides of (12)

$$V(z, y) - V(z_0, y_0) \leq \int_0^t v(\tau) y(\tau) d\tau - \lambda \int_0^t y^2(\tau) d\tau. \quad (13)$$

As $V(z, y) \geq 0$ let $\nu = V(z_0, y_0)$, then

$$\int_0^t v(\tau) y(\tau) d\tau + \mu \geq \lambda \int_0^t y^2(\tau) d\tau + V(z, y) \geq \lambda \int_0^t y^2(\tau) d\tau = \lambda \int_0^t y^2(\tau) d\tau.$$

it satisfies the passive definition (2b). Therefore, the Newton-Leipnik system (5) is rendered to be output strict passive (OSP) under the feedback control (11).

**Lemma 1** [28] If system (1) is made passive, then there exists a scalar storage function $V : \mathbb{R}^n \to \mathbb{R}$, $V(x) \geq 0, \forall x \neq 0$, and $S(x) \geq 0$ such that

$$L_y V(x) = -S(x)$$

$$L_x V(x) = h^T(x). \quad (14)$$

Where $L_y V(x) = \frac{\partial V^T}{\partial x} f(x)$ and $L_x V(x) = \frac{\partial V}{\partial x} g(x)$.

In order to explain Lemma 1 and also make ourselves clear enough, we take Newton-Leipnik as an example. Newton-Leipnik system (5) with control action (11) in its third equation can be turned into (15).

$$\dot{x}_1 = -\gamma x_1 + x_2 + 10x_2 x_3,$$

$$\dot{x}_2 = -x_2 - 0.4 x_1 + 5x_1 x_3,$$

$$\dot{x}_3 = -15x_1x_2 - \lambda x_3 - \varphi(x_3). \quad (15)$$

Notice that we replace $z_1, z_2$ and $y$ with $x_1, x_2$ and $x_3$. Consider Lyapunov function

$$V(x) = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2),$$

then
\[ L_y V(x) = \frac{\partial V^T}{\partial x} f(x) = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} -\gamma x_1 + x_2 + 10 x_1 x_3 \\ -x_1 - 0.4 x_2 + 5 x_1 x_3 \\ -15 x_1 x_2 - \lambda x_2 - x_1 \phi(x_1) \end{bmatrix} = -\gamma x_1^2 - 0.4 x_2^2 - \lambda x_2^2 - x_1 \phi(x_1) = -S(x) \]

and \[ L_y V(x) = \frac{\partial V^T}{\partial x} g(x) = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_3 = h^T(x). \] And in this way, it implies Lemma 1.

### 3 Coexisting attractors network

In this section, we take each single coexisting attractor dynamic model (which has been made passive) as an agent, and each agent dynamics can be written for \( i = 1, \ldots, N \) as

\[
\begin{align*}
\dot{x}_i &= f_i(x_i) + g_i(x_i) \bar{p}_i, \\
y_i &= h_i(x_i).
\end{align*}
\] (16)

**Assumption 1** Let \( G \) represents the network graph which is founded by the coexisting attractors (17), \( G \) is a strongly connected and balanced directed graph.

Let \( B = (b_{ij})_{N \times N} \) denotes the inter-coupling among the nodes of the coexisting attractor network. In this model, it is required that the coupling coefficients satisfy \( \sum_{j=1}^{N} b_{ij} = 0 \), this guarantees that the coupling is diffusive. If there is a connection between node \( i \) and node \( j \) (\( j \neq i \)), then \( b_{ij} = b_{ji} = 1 \); otherwise \( b_{ij} = b_{ji} = 0 \). According to Assumption 1, for \( G \) is connected, then \( B = B^T \) is an irreducible real symmetric matrix.

**Definition 2** Suppose that we have a network with \( N \) identical agents as (17). In the absence of communication delays, the agents are said to be output synchronized if

\[
\lim_{t \to \infty} \left| y_i(t) - y_j(t) \right| = 0 \quad \forall i, j = 1, \ldots, N
\] (17)

We analyze the case when the communication topology is fixed, i.e., the information graph does not change with time. Suppose that the agents are coupled using the control

\[
\bar{u}_i = \sum_{j \in N_i} (y_j - y_i), \forall i, j = 1, \ldots, N
\] (18)

Where \( N_i \) is the set of agents transmitting their outputs to the \( i^{th} \) agent.
Theorem 1 Consider the dynamical system described by (16) with control (18). Then under Assumption 1, the nonlinear system (16) will be globally stabilized by (18) and the agents outputs can be synchronized.

Proof. Consider a positive definite Lyapunov function for the $N$ agent system as

$$V = 2(V_1 + V_2 + \cdots + V_N)$$

(19)

Where $V_i$ is the storage function for agent $i$. The derivative of this Lyapunov function along trajectories of (16) is

$$\dot{V} = 2\sum_{i=1}^{N} (L_i V_i + (L_i V_i)\pi_i) = 2\sum_{i=1}^{N} (-S_i(x_i) + y_i^T \pi_i)$$

$$= -2\sum_{i=1}^{N} S_i(x_i) + 2\sum_{i=1}^{N} \sum_{j \in N_i} y_i^T (y_j - y_i)$$

$$= -2\sum_{i=1}^{N} S_i(x_i) - 2\sum_{i=1}^{N} \sum_{j \in N_i} y_i^T y_j + 2\sum_{i=1}^{N} \sum_{j \in N_i} y_j^T y_i$$

(20)

As the information exchange graph is balanced, we have

$$2\sum_{i=1}^{N} \sum_{j \in N_i} y_i^T y_j = \sum_{i=1}^{N} \sum_{j \in N_i} y_i^T y_j + \sum_{i=1}^{N} \sum_{j \in N_i} y_j^T y_i$$

(21)

And, therefore, it follows that

$$\dot{V} = -2\sum_{i=1}^{N} S_i(x_i) - \sum_{i=1}^{N} \sum_{j \in N_i} (y_i - y_j)^T (y_i - y_j) \leq 0$$

(22)

Thus the system is globally stable and all signals are bounded. Consider the set $E = \{ x_i \in \mathbb{R}^{n}, i = 1, \ldots, N | \dot{V} = 0 \}$. The set $E$ is characterized by all trajectories such that

$$\{ S_i(x_i) = 0, (y_i - y_j)^T (y_i - y_j) = 0, \forall j \in N_i, \forall i = 1, \ldots, N \}.$$ Lasalle’s Invariance Principle\cite{29} and strong connectivity of the network then implies output synchronization of (16).

4 Examples and research results

In order to further explain the hierarchical control mechanism, three examples for network topologies beginning with star connection, passing from ring connection and extending to chain connection are provided.

4.1 Star connected

The star connected coexisting attractor network is as Fig.2, the nodes are labeled from 1 to 5, respectively. Here $N = 5$. 

\[ \text{8} \]}
The inter-coupling matrix \( B \) is

\[
\begin{bmatrix}
-4 & 1 & 1 & 1 \\
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

(23)

So, the corresponding inter control action of each nodes can be listed as follows

\[
\begin{align*}
\overline{u}_1 &= \sum_{j \in N_i} (y_j - y_i), \\
&= (y_2 - y_i) + (y_3 - y_i) + (y_4 - y_i) + (y_5 - y_i), \\
\overline{u}_2 &= \sum_{j \in N_i} (y_j - y_2), \\
&= y_1 - y_2, \\
\overline{u}_3 &= \sum_{j \in N_i} (y_j - y_3), \\
&= y_1 - y_3, \\
\overline{u}_4 &= \sum_{j \in N_i} (y_j - y_4), \\
&= y_1 - y_4, \\
\overline{u}_5 &= \sum_{j \in N_i} (y_j - y_5), \\
&= y_1 - y_5.
\end{align*}
\]

(24)

The intra control action of each nodes can be expressed as

\[
\begin{align*}
u_i &= -\sigma (y_i - 10z_i z_2 - \lambda y_i) + v_i, i = 1, ..., 5, \\
&= -\sigma x_{1i} - 10x_{2i} x_{12} - \lambda x_{3i} + v_i.
\end{align*}
\]

(25)

Where \( \lambda \) is uniformly selected as 2 in this paper. Output synchronization at the manifold of origin is listed in Fig.3.

The initial outputs of the coexisting agents are

\[
[y_{10} \ y_{20} \ y_{30} \ y_{40} \ y_{50}]^T = [-0.5 \ -0.2 \ 0.1 \ 0.2 \ 0.4]^T.
\]

If we want to drive the coexisting network synchronize at the other non-zero manifold, extra reference input signal must be injected in, the injected input reference signal acted as targeting output shift, for example with extra injecting input reference...
constant signal $y_c = 0.5310$, the coexisting networks can be synchronized at the manifold of 0.2103, this corresponding to the third coordinate or the eye centers of the upper attractor$^{[23]}$, referring to Fig.4.

4.2 Ring connected

The ring connected coexisting attractor network is as Fig.5. For simplicity, we only leave the inter coupling-matrix and the research results there.

\[
B = \begin{pmatrix}
-2 & 1 & 0 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 1 & -2
\end{pmatrix}
\] (26)

The corresponding output synchronization results are as Fig.6 and Fig.7, respectively.

4.3 Chain connected

The chain connected coexisting attractor network is as Fig.8. And for the same reason like ring connected case, we only leave the inter coupling-matrix and the research results.

Fig.4 Output synchronization at the manifold of 0.2103 of star connected network

Fig.5 Ring connected coexisting attractors network

Fig.6 Output synchronization at the manifold of origin of ring connected network

Fig.7 Output synchronization at the manifold of 0.2103 of ring connected network

Fig.8 Chain connected coexisting attractors network
The corresponding output synchronization results are as Fig.9 and Fig.10, respectively.

\[
B = \begin{bmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]  

(27)

5 Conclusions

Coexisting attractor networks output synchronization is discussed in this paper. Unlike the already existing results, a hierarchical control strategy is used here, which can be divided into two different parts, one is the intra control, aiming at node dynamics passive realization and the other one is the inter control, which works in a distributed way, emphasizing on coordination of the agents. Hierarchical control can be equivalently looked upon as the synergy of intra control and inter control. Coexisting network output synchronization is the natural evolvement under the coordination strategy of hierarchical control. For its free from linearization, the results are global, which is vital important for nonlinear systems, and besides that method here pays attention to both nodes dynamics and network topology, which are the two fundamental factors of networks, for owing of that, it paves the way for harmonious manipulation between the network and its nodes in some extent.

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Fig. 1 The double strange attractors of the Newton-Leipnik system
Fig.2 Star connected coexisting attractors network
Fig. 3 Output synchronization at the manifold of origin of star connected network
Fig. 4 Output synchronization at the manifold of 0.2103 of star connected network
Fig. 5 Ring connected coexisting attractors network
Fig. 6 Output synchronization at the manifold of origin of ring connected network
Fig. 7 Output synchronization at the manifold of 0.2103 of ring connected network
Fig. 8 Chain connected coexisting attractors network
Fig. 9 Output synchronization at the manifold of origin of chain connected network
Fig. 10 Output synchronization at the manifold of 0.2103 of chain connected network