Gradient Flow Based Semi-implicit Finite Element Method and its Convergence Analysis for Image Reconstruction

Chong Chen
LSEC, ICMSEC, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China
Email: chench@lsec.cc.ac.cn

Guo-Liang Xu
LSEC, ICMSEC, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China
Email: xuguo@lsec.cc.ac.cn

Abstract

In this paper, we present a novel $L^2$-gradient flow based semi-implicit finite element method for solving a variational model of image reconstruction under various data scenarios, especially for the contaminated data detected from uniformly few or randomly distributed projection angles. We give a rigorous proof for the convergence of the semi-implicit finite element method. Finally, various numerical experiments are presented which demonstrate that our method is stable and effective.

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1. Introduction

The image reconstruction in computed tomography (CT), or some other application fields, for instance, electron tomography (ET), electron microscopy (EM), astronomy, geophysics, is to produce an image from a large number of its line-integral projections from different directions. In many applications of tomography, the reconstructed image reflects some kind of ray attenuation coefficient when it travels through the detected object, and the line integrals are obtained by measuring the attenuation of photons transmitting through the measured object. It is typically an inverse problem when we use the observed data though certain deterministic systems to inverse the degree of attenuation or some kind of physical quantity in actually applications.

On the one hand, the filtered back-projection (FBP) algorithm has been firstly introduced in the medical field by [21, 23], and in the radio astronomy by [2]. The FBP, one pivotal component of commercial CT scanners, has remained popular for the past 25 years [19]. Main disadvantages of FBP, however, lie in the lack of anti-noise when the detected data are contaminated by random noise and produce acute artifacts if the projection views are uniformly very few or distributed randomly. These drawbacks are demonstrated by our numerical illustrations in Section 5 of this paper. The direct Fourier reconstruction (DF), in addition, has been investigated in [7,17,20,22]. Generally speaking, the performance of FBP is better than that of DF since the former yields reconstructed images with better quality [14].

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Algebraic based approaches for image reconstruction, on the other hand, have been extensively studied, such as algebraic reconstruction techniques (ART) [12], simultaneous iterative reconstructive technique (SIRT) [11], simultaneous algebraic reconstruction technique (SART) [1], and so forth. These techniques can be implemented readily in the actual reconstructive process, although they are deficient in stability and usually suffer from random noise which is caused by the inconsistencies introduced in a set of the equations by the approximations commonly used for the coefficient elements [15]. Optimization based regularized algorithm, therefore, has begun to be in fashion. The general form of optimization based algorithm can be formulated as follows:

$$\hat{f}^* = \arg \min_f \left( \| A\hat{f} - \tilde{g} \|_w^2 + \alpha \mathcal{R}(\tilde{f}) \right),$$

where the elements in vectors $\tilde{g}$ and $\tilde{f}$ of finite dimensions denote data measurements and basis coefficients associated with the image-pixel or voxel values, $A$ stands for the coefficient matrix, $\| \cdot \|_w$ is the standard weighted Euclidean norm, the first term in the brackets of the right side is the fidelity to the observed data, $\mathcal{R}(\tilde{f})$ and $\alpha$ express the regularization functional and regularization parameter [3,6,19,24].

Recently, Guoliang Xu et al. proposed a new variational model for ET image reconstruction and solved it using $L^2$-gradient flow based explicit finite element method [27]. Furthermore, Chong Chen et al. modified this variational model for image reconstruction with contaminated data detected from uniformly few or randomly distributed projection angles and solved it by $L^2$-gradient flow based explicit finite difference method [4]. These two papers, however, did not give the convergence analysis of the correspondingly computational methods. Furthermore, these two computational methods are sensitive to the choice of temporal step size due to the using of the explicit element method. In this paper, we present a new $L^2$-gradient flow based semi-implicit finite element method for solving the variational model of image reconstruction under contaminated data detected from uniformly few or randomly distributed projection angles. In addition, we also give a rigorous proof for the convergence of the semi-implicit finite element method. Finally, various numerical experiments are presented which demonstrate that our method is stable and effective.

The outline of this paper goes as follows. Section 2 sketches an overview of the mathematical background knowledge on image reconstruction. In Section 3, we come up with a variational model, then propose an $L^2$-gradient flow based semi-implicit finite element method for solving it, and ultimately give the concrete numerically computational procedures. The complete theoretical analysis of our numerical method is given in Section 4. In Section 5 numerical results under various data scenarios are presented. Finally, Section 6 concludes the paper.

2. Mathematical Preliminaries

The purpose of this section is to present various integral transforms and some their important properties. Moreover, explanation of notations are also presented here. The materials of this section serve as the background knowledge for the rest of this paper. For the detail derivation, we suggest the interested readers referring to [9,13,18].

Let $f$ be a function defined on $\mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional (n-D) real space consisting of n-tuples of real numbers, usually denoted by single letters, $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, etc. The inner product and norm in $\mathbb{R}^n$ are defined as $\langle x, y \rangle = x^T y = \sum_1^n x_i y_i$ and $\| x \| = \sqrt{\langle x, x \rangle}$, respectively. In addition, the gradient of $f$ is denoted as $\nabla f =$
also denoted as $R$. Where $u$ where $h$.

As shown in (2.2), we can see that the X-ray transform is the integral of $f$ over the unit cylinder $Z = S^{n-1} \times R^1$ of $R^{n+1}$. Specifically, in what follows we have its analytic formulation

$$ R f(\theta, s) = \int_{(x, \theta) = s} f(x) dx $$

where the integrand $f \in L^2(R^n)$, the Lebesgue square integrable space, $\theta \in S^{n-1}$, the unit sphere in $R^n$, $s \in R^1$, the distance from the origin to the hyperplane perpendicular to $\theta$, $\Theta^\perp$ is the hyperplane passing through the origin and orthogonal to $\theta$. In formula (2.1), $R f(\theta, s)$ is also denoted as $R_\theta f(s)$.

The X-ray transform of a function $f \in L^2(R^n)$ is defined as

$$ P f(\theta, y) = P_\theta f(y) = \int_{R^1} f(s\theta + y) ds, \quad y \in \Theta^\perp. \quad (2.2) $$

As shown in (2.2), we can see that the X-ray transform is the integral of $f \in L^2(R^n)$ over the straight line through point $y \in \Theta^\perp$ along the direction $\theta \in S^{n-1}$. Hence we can regard $P f$ as a function defined on the tangent bundle

$$ T = \{(\theta, y) : \theta \in S^{n-1}, \ y \in \Theta^\perp\}. $$

Consider the 2-D case. Let $\theta = (\cos \phi, \ sin \phi)^T$, then $\theta^\perp = (-\sin \phi, \ cos \phi)^T$. Therefore $\Theta^\perp = \{s\theta^\perp : s \in R^1\}$ and we have the following equation

$$ P f(\theta, s\theta^\perp) = R f(\theta^\perp, s). $$

Hence for the sake of consistency in $R^n$ tomography we prefer to use X-ray transform form for all integer $n \geq 2$.

The inner product in $L^2(R^n)$ is defined as

$$ \langle u(x), v(x) \rangle = \int_{R^n} u(x)v(x)dx, $$

where $u(x), v(x) \in L^2(R^n)$. The inner product in $L^2(T)$ is given by

$$ \langle h_1(\theta, y), h_2(\theta, y) \rangle = \int_T h_1(\theta, y)h_2(\theta, y)d\theta dy, $$

where $h_1(\theta, y), h_2(\theta, y) \in L^2(T)$.

Using the definition of $P_\theta$, we obtain the following lemma.
Lemma 2.1. Let \( \mathcal{O} \subset \mathbb{R}^n \) be a sphere with radius \( R \). \( R \) is sufficient large but finite. Then the projection operator

\[
P_\theta : L^2(\mathcal{O}) \rightarrow L^2(\Theta^\perp)
\]

is linear and continuous.

Proof. According to the definition of the projection operator, we have

\[
P_\theta f(y) = \int_{\mathbb{R}^1} f(y + s\theta) ds,
\]

where \( f(x) \in L^2(\mathbb{R}^n) \) with finite support in \( \mathcal{O} \). Hence, the linear property is valid obviously.

Using the Cauchy-Schwartz inequality, we have

\[
|P_\theta f(y)|^2 = \left| \int_{\mathbb{R}^1} f(y + s\theta) ds \right|^2 \\
\leq C \int_{\mathbb{R}^1} |f(y + s\theta)|^2 ds,
\]

where \( C \) is a positive constant. Then,

\[
\int_{\Theta^\perp} |P_\theta f(y)|^2 dy \leq C \int_{\Theta^\perp} \int_{\mathbb{R}^1} |f(y + s\theta)|^2 ds dy \\
= C \int_{\mathbb{R}^n} |f(x)|^2 dx.
\]

Therefore the projection operator is continuous. This completes the proof of the lemma.

Integrating over \( S^{n-1} \) on the two sides of inequality (2.3) gives the linear continuity of \( P \). Similar proofs of linearly continuous property of \( R_\theta \) and \( R \) can be found in [18]. Basing upon the definition and property of \( P \), we can obtain its corresponding adjoint operator \( P^* \). According to the following definition

\[
\langle Pf(\theta, y), h(\theta, y) \rangle = \langle f(x), P^* h(x) \rangle,
\]

we have

\[
P^* h(x) = \int_{S^{n-1}} h(\theta, x - \langle x, \theta \rangle \theta) d\theta.
\]

Theorem 2.1. For \( f \in L^2(\mathbb{R}^n) \) with compact support \( \Omega \), we have

\[
P^* P f = 2 \int_{\mathbb{R}^n} \|x - y\|^{1-n} f(y) dy.
\]

The detail proof of this theorem can be found in [18].

3. Gradient Flow Based Semi-implicit Finite Element Method

In many biomedical imaging applications, the detected objects usually locate in a finite region. Without loss of generality, we may assume that the image (object) to be reconstructed in 2-D or 3-D is restricted in a square or cube domain. In this paper, we mainly concentrate on the 2-D image reconstruction from the parallel projections from different angles because the reconstruction method proposed by us can be straightforwardly generalized to other projection geometries and higher dimensions.
3.1. Variational Model

Unlike the conventionally optimization based regularization model (1.1) that is based upon a discrete form, the variational model considered here is in a continuous framework. Let $f(x) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a density function of a cross section, which has a support in a bounded region $\Omega$, namely,

$$\text{supp}(f) \subseteq \Omega. \quad (3.1)$$

For simplicity, we assume that the imaging process is obtained through a linear and space-invariant system. Owing to the existence of noise, we will consider the effect of noise in the projection data observed from certain detected object. Therefore, in what follows we construct a general imaging model

$$g(\theta, y) = Pf(\theta, y) + n(\theta, y), \quad (3.2)$$

where $P$ represents the X-ray transform, $n(\theta, y)$ is assumed to be the additive random noise, and $g(\theta, y)$ stands for the observed data corrupted by noise $n(\theta, y)$. Hence we wish to reconstruct $f$ from the detected data $g$. The anisotropic variational model for the reconstruction can be formulated as the following minimization problem

$$\tilde{f} = \text{argmin}_{f \in \text{BV}(\Omega)} \left( E_1(f) + \lambda E_2(f) \right), \quad (3.3)$$

where

$$E_1(f) = \frac{1}{2} \int_T \left( Pf(\theta, y) - g(\theta, y) \right)^2 d\theta dy, \quad (3.4)$$

$$E_2(f) = \int_{\mathbb{R}^n} \phi(\|\nabla f\|) dx, \quad (3.5)$$

that is, $E_1(f)$ stands for the fidelity term to the observed data, $E_2(f)$ stands for the regularized term obtained from some maximum a posterior estimation or some significant operations, $\lambda$ is a positive parameter, balancing the effects of the fidelity term and the regularized one, $\phi$ is a properly chosen function satisfying some properties, and $\text{BV}(\Omega)$ stands for the bounded variation function space defined on $\Omega$. For the definition and properties of $\text{BV}(\Omega)$, we suggest the interested readers referring to [8].

The way on how to choose the potential function $\phi$ can be found in [3]. Function $\phi$ is the engine to remove interfered noise as well as to preserve geometric features. To solve the minimization problem (3.3), we need to compute the first order variation of the functional

$$E(f) = E_1(f) + \lambda E_2(f).$$

Then the Euler-Lagrange equations associated with functional $E(f)$ are

$$\left\{ \begin{array}{l}
\mathbf{P}^* \mathbf{P} f - \mathbf{P}^* g - \lambda \text{div} \left( \frac{\phi(\|\nabla f\|) \nabla f}{\|\nabla f\|} \right) = 0, \quad \text{in} \; \Omega, \\
\frac{\partial f}{\partial n} = 0, \quad \text{on} \; \text{the boundary of} \; \Omega = \partial \Omega,
\end{array} \right. \quad (3.6)$$

where $n$ is the outward normal of the boundary $\partial \Omega$.

From a theoretical point of view, the minimization model (3.3) admits an unique solution. We present this result in the following theorem.
**Theorem 3.1.** Assuming that \( \phi \) is a strictly convex, nondecreasing function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \), \( \lim_{s \to +\infty} \phi(s) = +\infty \). And there exists two constants \( c > 0 \) and \( b \geq 0 \) such that \( c s - b \leq \phi(s) \leq cs + b, \forall s \geq 0 \). Let \( P : L^2(\Omega) \to L^2(T) \) be the X-ray transform. Then the minimization problem
\[
\min_{f \in H^2(\Omega)} \left( E_1(f) + \lambda E_2(f) \right),
\]
where \( E_1 \) and \( E_2 \) are given by (3.4) and (3.5), respectively, admits a unique solution.

According to Lemma 2.1, we know that \( P : L^2(\Omega) \to L^2(T) \) is a linear continuous operator and \( P1_{\Omega} \neq 0 \) where \( 1_{\Omega} \) denotes any constant function defined on \( \Omega \). Therefore, the proof of existence and uniqueness of a solution for the minimization problem is similar to that of [26]. Hence, we do not give the proof because of nonessential difference.

### 3.2. Numerical Computing

In order to solve equations (3.6), we resort to gradient flow, i.e., convert the elliptic partial differential equation into a time-dependent parabolic one in the domain \([0, T] \times \Omega\), \( T \gg 0 \). When the parabolic partial differential equation (PDE) achieves its steady state solution, we immediately obtain the solution of Euler-Lagrange equations. Therefore, in what follows we solve
\[
\begin{cases}
\frac{\partial f}{\partial t} = \lambda \text{div} \left( \frac{\phi'(\|\nabla f\|)}{\|\nabla f\|} \right) - P^*Pf + P^*g, & \text{in } \Omega_{T_0} := (0, T_0] \times \Omega, \\
\frac{\partial f}{\partial n} = 0, & \text{on } \partial \Omega_{T_0} := (0, T_0] \times \partial \Omega,
\end{cases}
\tag{3.7}
\]
with given initial condition \( f_0 = f(x, y, 0) \). To preserve the geometric features of the reconstructed image as well as removing the artifacts and noise, we need to choose a proper regularization function \( \phi(s) \). Here we choose
\[
\phi(s) = \sqrt{s^2 + \epsilon^2}, \quad s \geq 0, \quad 0 \leq \epsilon < 1.
\tag{3.8}
\]
Since the first equation in (3.7) is not well defined at the points where \( \|\nabla f\| = 0 \) when choose \( \phi(s) = s \), namely total variation (TV) functional, we take a perturbation of TV as (3.8). For simplicity, we just consider numerical computation in 2-D situation because our method can be straightforwardly generalized to that of 3-D. Let
\[
E_{\lambda, \epsilon}(f) = E_1(f) + \lambda E_2(f).
\tag{3.9}
\]
For the purpose of solving gradient flow (3.7), we use a semi-implicit finite element method. Let \( V_h \) be the finite element space consisting of tensor product of the cube B-spline functions, \( h \in (0, 1) \) be the uniform step size of the spline grid, and \( v_h \) be the cube B-spline basis function in tensor product form, represented by \( B_k(x)B_l(y), k, l = 0, 1, \cdots, n - 1 \), where \( n \) depends on the displaying grid of the image to be reconstructed. Let \( \tau \in (0, 1) \) be the uniformly temporal step size of interval \([t_{m-1}, t_m]\) for \( m_0 \) equidistant partitions of \([0, T_0] \). We use forward Euler scheme with respect to the temporal direction, namely,
\[
\frac{\partial F^m}{\partial t} \approx d_t F^m := \frac{F^m - F^{m-1}}{\tau}.
\]
Then the semi-implicit finite element discretization for the gradient flow (3.7) is given as follows: Find \( F^m \in V_h \) for \( m = 1, 2, \cdots, m_0 \) such that
\[
\int_{\Omega} \left[ d_t F^m v_h + \lambda \frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|} \nabla F^m T \nabla v_h + (P^*P F^m - P^*g) v_h \right] \, dx = 0, \quad \forall v_h \in V_h,
\tag{3.10}
\]
with an initial function $F^0 \in V_h$ that approximates to $f_0$.

**Remark 3.1.** The novelty of our formulation is that we use $\int_{\mathbb{R}^n}(P^*Pf^m - P^*g)v_hdx$ instead of $\sum_i \int_{\mathbb{R}^n-1} (Pf(\theta, y) - g(\theta, y))Pv_h(\theta, y)dy$ (see [27]), where $P^*P$, given by (2.5), does not depend on the projection. This new formulation makes our method effective even for projections of sparse angles.

**Remark 3.2.** In [10], the Remark 1.6 is also valid here for our semi-implicit scheme. In addition, Xiaobing Feng et al proposed an implicit finite element method for variational model on image denoising [10]. In their discrete scheme, the $P$ is merely an identity operator. From the computational point of view, the implicit finite element method is inconvenient for numerical computation due to its nonlinear property. Here our semi-implicit finite element method, however, is linear so that numerical calculation is feasible. Our method is also applicable for image denoising.

For simplicity, several notations need to be introduced. Let

$$F^m(x, y) = \sum_{i,j} f_N^m B_i(x) B_j(y),$$

$$B_{MN} = \int_\Omega B_i B_j B_k B_l dx,$$

$$P_{MN} = \int_\Omega P^* P(B_i B_j) B_k B_l dx = \int_\Omega P(B_i B_j) P(B_k B_l) d\theta dy,$$

$$G_M = \int_\Omega P^* (g) B_k B_l d\theta dx = \int_\Omega g P(B_k B_l) d\theta dy,$$

$$Q_{MN}^{m-1} = \int_\Omega \left( \frac{\phi(\|F^{m-1}\|)}{\|F^{m-1}\|} \right) \nabla (B_i B_j) \nabla (B_k B_l) dx,$$

where $M = kn + l + 1, N = in + j + 1, i, j, k, l = 0, 1, \cdots, n - 1$, and $B_i B_j, B_k B_l$ are the abbreviated versions for $B_i(x) B_j(y), B_k(x) B_l(y)$, respectively. Let vectors $f^m = (f_N^m)_{n^2 \times 1}$, $G = (G_M)_{n^2 \times 1}$, matrices $B = (B_{MN})_{n^2 \times n^2}, P = (P_{MN})_{n^2 \times n^2}$, and $Q^{m-1} = (Q_{MN}^{m-1})_{n^2 \times n^2}$.

**Remark 3.3.** According to the definition of Gram matrix, $B, P, Q^{m-1}$ are all Gram matrices. Note that the basis functions $\{B_i B_j\}$ and $\{P(B_i B_j)\}, i, j = 0, 1, \cdots, n - 1$ are linear independence but $\{\nabla(B_i B_j)\}$ may be linear dependence. Therefore, $B$ and $P$ are both positive definite matrices yet $Q^{m-1}$ is at least positive semi-definite.

Taking $v_h = B_k(x) B_l(y), k, l = 0, 1, \cdots, n - 1$ and using (3.11), formula (3.10) becomes

$$\sum_N B_{MN} f_N^{m+1} = \sum_N B_{MN} f_N^m + \tau \left( G_M - \sum_N P_{MN} f_N^{m+1} - \lambda \sum_N Q_{MN}^{m+1} f_N^{m+1} \right),$$

where $m = 0, 1, \cdots, m_0 - 1$. Using matrix notations, we further obtain the following iterative scheme

$$(B + \tau(P + \lambda Q^{m})) f^{m+1} = B f^m + \tau G,$$

where $B + \tau(P + \lambda Q^{m})$, on the left side of iterative scheme (3.17), is a positive definite matrix as a result of Remark 3.3. Hence generalized minimal residual method (GMRES) can be used to solve (3.17) efficiently. In what follows we present our image reconstruction algorithm under the semi-implicit finite element discretization.
Algorithm 3.1 (Semi-implicit Finite Element Method)

Step 1 Given an initial point \( f^0 \in \mathbb{R}^{n^2} \), \( 0 \leq \alpha \ll 1 \), and an integer \( K > 0 \).

Step 2 Set \( k := 0 \).

Step 3 Evaluate \( f^{k+1} \) by GMRES under iterative scheme (3.17), namely, \( f^{k+1} = \text{GMRES}(f^k) \), and compute error \( r_k = |E_{\lambda,\epsilon}(f^{k+1}) - E_{\lambda,\epsilon}(f^k)| \).

Step 4 If \( r_k \leq \alpha \) or \( k > K \), stop iteration, otherwise set \( k := k + 1 \), return to Step 3.

According to the property of finite support of B-spline basis function, the matrices \( B \) and \( Q^m \) are sparse. The matrix \( P \), however, is dense. Hence, the entire \( B^\tau + \tau(P + \lambda Q^m) \) is a dense matrix. If the size of \( n \) is large, the required memory for storing \( B^\tau + \tau(P + \lambda Q^m) \) may beyond the capacity of the used computer. Thus it is adverse to compute the coefficient matrix directly in advance. To overcome this shortcoming, we compute \( (B^\tau + \tau(P + \lambda Q^m))f^m \) for GMRES with initial point \( f^m \) to obtain \( f^{m+1} \). Note that

\[
\sum_{N} P_{MN} f^m_N = \int_{T} P F^m(B_k B_l) d\theta dy. \quad (3.18)
\]

By the fact of Theorem 2.1, formula (3.18) becomes

\[
\sum_{N} P_{MN} f^m_N = 2 \int_{\Omega} \int_{\mathbb{R}^2} \|x - y\|^{-1} F^m dy B_k B_l dx. \quad (3.19)
\]

In order to speed up the computing, the fast Fourier transform (FFT) can be used to compute the part of convolution.

4. Convergence Analysis of Semi-implicit Finite Element Method

In this section, we give the convergent analysis of finite element discretization for semi-implicit scheme. For each iterative finite element solution \( \{F^m\} \), in what follows we give its constant and linear interpolation in temporal direction \( t \) as that of [10]

\[
\overline{F}^{\epsilon,h,\tau}(\cdot, t) := F^{m-1}, \quad \forall t \in [t_{m-1}, t_m], \quad 1 \leq m \leq m_0, \quad (4.1)
\]

\[
\overline{F}^{h,\tau}(\cdot, t) := \frac{t - t_{m-1}}{\tau} F^m + \frac{t_m - t}{\tau} F^{m-1}, \quad \forall t \in [t_{m-1}, t_m], \quad 1 \leq m \leq m_0. \quad (4.2)
\]

Obviously, \( \overline{F}^{\epsilon,h,\tau} \) is continuous in spatial \( x \) but discontinuous in time \( t \). However, \( \overline{F}^{h,\tau} \) is continuous in both \( x \) and \( t \).

Theorem 4.1. Assume that \( f_0 \in L^2(\Omega) \), \( g \in L^2(T) \) and \( \partial\Omega \) is sufficiently regular. Then, for each fixed \( \epsilon > 0 \), \( \{F^m\}_{m=1}^{m_0} \) derived from semi-implicit finite element scheme (3.10) such that

\[
\tau \sum_{m=1}^{q} \left[ \|d_t F^m\|_{L^2}^2 + \frac{\tau}{2} \|d_t (P F^m - g)\|_{L^2}^2 \right] + E_{\lambda,\epsilon}(F^q) \leq E_{\lambda,\epsilon}(F^0), \quad 1 \leq q \leq m_0. \quad (4.3)
\]

In addition, under the following initial condition constraint

\[
\lim_{h \to 0} \|f_0 - F^0\|_{L^2} = 0,
\]
there exists a unique $f^* \in L^\infty((0, T_0); \text{BV}(\Omega)) \cap H^1((0, T_0); L^2(\Omega))$ such that

$$\lim_{h, \tau \to 0} \| f^* - F^{e, h, \tau} \|_{L^\infty((0, T_0); L^p(\Omega))} = 0, \quad (4.4)$$

$$\lim_{h, \tau \to 0} \| f^* - \overline{F}^{e, h, \tau} \|_{L^\infty((0, T_0); L^p(\Omega))} = 0, \quad (4.5)$$

for any $p \in \left[1, \frac{n}{n-1}\right)$.

**Proof.** To verify (4.3), using test function $v_h = d_t F^m$ for (3.10), we have

$$\|d_t F^m\|_{L^2}^2 + \int_{\Omega} \left[ \lambda \phi'(|\nabla \psi| - 1) F^m \cdot \nabla d_t F^m + (P^* P F^m - P^* g) d_t F^m \right] dx = 0. \quad (4.6)$$

Considering the third term on the left hand of (4.6), we derive

$$\int_{\Omega} (P^* P F^m - P^* g) d_t F^m dx = \frac{d_t \| P F^m - g \|_{L^2}^2}{2} + \frac{\tau \| d_t (P F^m - g) \|_{L^2}^2}{2}, \quad (4.7)$$

and similarly,

$$\nabla F^m \cdot \nabla d_t F^m = \frac{d_t \| \nabla F^m \|_{L^2}^2 + \tau \| \nabla d_t F^m \|_{L^2}^2}{2}. \quad (4.8)$$

Hence, as a result of (4.7) and (4.8), formula (4.6) becomes

$$\|d_t F^m\|_{L^2}^2 + \frac{d_t \| P F^m - g \|_{L^2}^2}{2} + \frac{\tau \| d_t (P F^m - g) \|_{L^2}^2}{2}$$

$$+ \frac{\lambda}{2} \int_{\Omega} \phi'(|\nabla \psi| - 1) \left( d_t \| \nabla F^m \|_{L^2}^2 + \tau \| \nabla d_t F^m \|_{L^2}^2 \right) dx = 0. \quad (4.9)$$

For the forth term on the left hand side of (4.9), we have

$$\frac{1}{2} \int_{\Omega} \phi'(|\nabla \psi| - 1) d_t \| \nabla F^m \|_{L^2}^2 dx = \frac{1}{\tau} \int_{\Omega} \phi'(|\nabla \psi| - 1) (\| \nabla F^m \| - \| \nabla F^m - 1 \|) dx$$

$$+ \frac{1}{2\tau} \int_{\Omega} \phi'(|\nabla \psi| - 1) \left( \| \nabla F^m \| - \| \nabla F^m - 1 \| \right)^2 dx. \quad (4.10)$$

Using Cauchy inequality $\nabla F^m \cdot \nabla F^m - \| \nabla F^m \| \| \nabla F^m - 1 \| \leq \| \nabla F^m \| \| \nabla F^m - 1 \|$, and $\phi'(s) \geq 0$, we obtain

$$\frac{1}{2\tau} \int_{\Omega} \phi'(|\nabla \psi| - 1) \left( \| \nabla F^m \| - \| \nabla F^m - 1 \| \right)^2 dx \leq \frac{\tau}{2} \int_{\Omega} \phi'(|\nabla \psi| - 1) \| \nabla d_t F^m \|_{L^2}^2 dx. \quad (4.11)$$

Substituting (4.10)-(4.11) into (4.9), we obtain

$$\|d_t F^m\|_{L^2}^2 + \frac{d_t \| P F^m - g \|_{L^2}^2}{2} + \frac{\tau \| d_t (P F^m - g) \|_{L^2}^2}{2}$$

$$+ \frac{\lambda}{\tau} \int_{\Omega} \phi'(|\nabla \psi| - 1) (\| \nabla F^m \| - \| \nabla F^m - 1 \|) dx$$

$$+ \frac{\lambda}{\tau} \int_{\Omega} \phi'(|\nabla \psi| - 1) \left( \| \nabla F^m \| - \| \nabla F^m - 1 \| \right)^2 dx \leq 0. \quad (4.12)$$
We now show that
\[
\int_{\Omega} \phi'(\|\nabla F^m\|)(\|\nabla F^m\| - \|\nabla F^{m-1}\|)dx + \int_{\Omega} \phi'(\|\nabla F^{m-1}\|)(\|\nabla F^m\| - \|\nabla F^{m-1}\|)^2 dx \\
\geq \int_{\Omega} \phi'(\|\nabla F^m\|)(\|\nabla F^m\| - \|\nabla F^{m-1}\|)dx \quad (4.13)
\]
\[
\geq \int_{\Omega} \phi(\|\nabla F^m\|) - \phi(\|\nabla F^{m-1}\|)dx. \quad (4.14)
\]

To show above inequalities, the integral domain \( \Omega \) is separated into two disjoint subsets, i.e.
\[
\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \emptyset,
\]
where \( \Omega_1 := \{ x \in \Omega : \|\nabla F^m\| \geq \|F^{m-1}\| \} \) and \( \Omega_2 := \{ x \in \Omega : \|\nabla F^m\| < \|F^{m-1}\| \} \). In \( \Omega_1 \), we have
\[
\frac{1}{\sqrt{\|\nabla F^{m-1}\|^2 + \epsilon^2}} \geq \frac{1}{\sqrt{\|\nabla F^m\|^2 + \epsilon^2}},
\]
namely,
\[
\frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|} \geq \frac{\phi'(\|\nabla F^m\|)}{\|\nabla F^m\|}.
\]
Then, we obtain
\[
\phi'(\|\nabla F^{m-1}\|)\|\nabla F^m\| \geq \phi'(\|\nabla F^m\|)\|\nabla F^{m-1}\|.
\]
Adding \(-\phi'(\|\nabla F^{m-1}\|)\|\nabla F^{m-1}\|\) onto two sides of above inequality, we have
\[
\phi'(\|\nabla F^{m-1}\|)(\|\nabla F^m\| - \|\nabla F^{m-1}\|) \geq (\phi'(\|\nabla F^m\|) - \phi'(\|\nabla F^{m-1}\|))\|\nabla F^{m-1}\|.
\]
Hence, we get
\[
\phi'(\|\nabla F^{m-1}\|)(\|\nabla F^m\| - \|\nabla F^{m-1}\|) \geq (\phi'(\|\nabla F^m\|) - \phi'(\|\nabla F^{m-1}\|))\|\nabla F^{m-1}\|.
\]
Since \(\|\nabla F^m\| - \|\nabla F^{m-1}\| \geq 0\) is valid in \(\Omega_1\), multiplying \(\|\nabla F^m\| - \|\nabla F^{m-1}\|\) on two sides of above inequality, we have
\[
\frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|}(\|\nabla F^m\| - \|\nabla F^{m-1}\|)^2 \geq (\phi'(\|\nabla F^m\|) - \phi'(\|\nabla F^{m-1}\|))\|\nabla F^m\| - \|\nabla F^{m-1}\|).
\]
Therefore, the following inequality is verified in \(\Omega_1\)
\[
\phi'(\|\nabla F^{m-1}\|)(\|\nabla F^{m-1}\| - \|\nabla F^m\|) + \frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|}(\|\nabla F^m\| - \|\nabla F^{m-1}\|)^2 \\
\geq \phi'(\|\nabla F^{m-1}\|)\|\nabla F^m\| - \|\nabla F^{m-1}\|). \quad (4.15)
\]
On the other hand, as a result of \(\|\nabla F^m\| < \|\nabla F^{m-1}\|\) in \(\Omega_2\), we have
\[
\frac{1}{\sqrt{\|\nabla F^{m-1}\|^2 + \epsilon^2}} < \frac{1}{\sqrt{\|\nabla F^m\|^2 + \epsilon^2}},
\]
namely,
\[
\frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|} < \frac{\phi'(\|\nabla F^m\|)}{\|\nabla F^m\|}.
\]
Then, we obtain
\[ \phi'(\|\nabla F^m\|)\|\nabla F^m\| \leq \phi'(\|\nabla F^m\|)\|\nabla F^m\|. \]
Adding \(-\phi'(\|\nabla F^m-\|\nabla F^m\|)\|\nabla F^m\|\) onto two sides of above inequality, we have
\[ \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \phi'(\|\nabla F^m\|)\|\nabla F^m\| \leq \phi'\left(\|\nabla F^m\| - \phi'(\|\nabla F^m\|)\|\nabla F^m\|\right). \]
Hence, we get
\[ \frac{\phi'(\|\nabla F^m\|)}{\|\nabla F^m\| - \|\nabla F^m\|} \leq \phi'(\|\nabla F^m\|) - \phi'(\|\nabla F^m\|). \]
Since \(\|\nabla F^m\| - \|\nabla F^m\| < 0\) is valid in \(\Omega_2\), multiplying \(\|\nabla F^m\| - \|\nabla F^m\|\) on two sides of above inequality, we have
\[ \frac{\phi'(\|\nabla F^m\|)}{\|\nabla F^m\| - \|\nabla F^m\|} \geq \phi'(\|\nabla F^m\|) - \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\|. \]
Therefore, the inequality (4.15) is also verified in \(\Omega_2\). Using (4.15) in \(\Omega\), we have
\[ \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\| \geq \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\|\|\nabla F^m\| - \|\nabla F^m\|. \]
Hence, integrating on two sides of (4.16) over \(\Omega\), we obtain
\[
\int_{\Omega} \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\|dx + \int_{\Omega} \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\|\|\nabla F^m\| - \|\nabla F^m\|dx \geq \int_{\Omega} \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\|dx.
\]
Therefore, (4.13) is proved.

By the convexity of \(\phi(s)\), the term on the right hand side of (4.17) is bounded by
\[
\int_{\Omega} \phi'(\|\nabla F^m\|)\|\nabla F^m\| - \|\nabla F^m\|dx \geq \tau dt \int_{\Omega} \phi(\|\nabla F^m\|)dx.
\]
Then (4.14) is obtained.

According to (4.12), (4.17) and (4.18), we get
\[
\|d F^m\|_2^2 + \frac{d|F^m - g|^2}{2} + \tau |d t(F^m - g)|_2^2 + \lambda dt \int_{\Omega} \phi(\|\nabla F^m\|)dx \leq 0.
\]
Applying the summation operator \(\tau \sum_{m=1}^{q} \) to the above inequality, we get (4.3).

To show (4.4)-(4.5), we first notice that (4.3) implies the following (uniform in both \(h, \tau\) and \(\epsilon\) estimates
\[
\|F_t^h,\tau\|_{L^2(\Omega)} = \left( \tau \sum_{m=1}^{q} \|d F^m\|_2^2 \right)^{\frac{1}{2}} \leq C,
\]
\[
\|P F_t^h,\tau\|_{L^2(\Omega)} \leq \|P F_t^h,\tau\|_{L^2(\Omega)} = \max_{0 \leq m \leq m_0} \|P F^m\|_2 \leq C,
\]
\[
\|\nabla F_t^h,\tau\|_{L^2(\Omega)} \leq \|\nabla F_t^h,\tau\|_{L^2(\Omega)} = \max_{0 \leq m \leq m_0} \|\nabla F^m\| \leq C, \text{ if } \lambda \neq 0,
\]
\[
\sum_{m=1}^{m_0} \|P F^m - P F^{m-1}\|_2^2 \leq C,
\]
where \( C \) is a sufficiently large constant.

Next, using test function \( v_h = F^m \) for (3.10), we obtain
\[
\int_\Omega \left[ d_t F^m \cdot F^m + \lambda \frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|} \nabla F^m \cdot \nabla F^m + P^*(P F^m - g) F^m \right] dx = 0. \tag{4.24}
\]

For the first and the third terms in the integration of (4.24), we derive
\[
d_t F^m \cdot F^m = \frac{F^m - F^{m-1}}{\tau} F^m = \frac{d_t |F^m|^2}{2} + \frac{\tau |d_t F^m|^2}{2}, \tag{4.25}
\]
and
\[
\int_\Omega P^*(P F^m - g) F^m dx = \int_\Gamma (P F^m - g) P F^m dT = \|P F^m - g\|_{L^2}^2 + \|P F^m\|_{L^2}^2 - \|g\|_{L^2}^2. \tag{4.26}
\]

Using (4.25) and (4.26), formula (4.24) becomes
\[
\frac{d_t \|F^m\|_{L^2}^2}{2} + \frac{\tau \|d_t F^m\|_{L^2}^2}{2} + \int_\Omega \lambda \frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|} \|\nabla F^m\|_{L^2}^2 dx \\
+ \frac{\|P F^m - g\|_{L^2}^2}{2} + \frac{\|P F^m\|_{L^2}^2}{2} = \frac{\|g\|_{L^2}^2}{2}. \tag{4.27}
\]

Therefore, applying the summation operator \( 2 \tau \sum_{m=1}^{q} \) to the resulting equality (4.27), we obtain
\[
\|F^q\|_{L^2}^2 + \tau \sum_{m=1}^{q} \|d_t F^m\|_{L^2}^2 + \tau \sum_{m=1}^{q} \int_\Omega \lambda \frac{\phi'(\|\nabla F^{m-1}\|)}{\|\nabla F^{m-1}\|} \|\nabla F^m\|_{L^2}^2 dx \\
+ \tau \sum_{m=1}^{q} \|P F^m - g\|_{L^2}^2 + \tau \sum_{m=1}^{q} \|P F^m\|_{L^2}^2 \leq \tau q \|g\|_{L^2}^2 + \|F^0\|_{L^2}^2, \quad \forall 1 \leq q \leq m_0. \tag{4.28}
\]

Since \( \tau q \leq T_0 \), we have
\[
\tau q \|g\|_{L^2}^2 + \|F^0\|_{L^2}^2 \leq T_0 \|g\|_{L^2}^2 + \|F^0\|_{L^2}^2 \leq C, \quad \forall 1 \leq q \leq m_0. \tag{4.29}
\]

Hence, according to (4.28) and (4.29), we get
\[
\|\overline{F}^{r,h,\tau}\|_{L^\infty(L^2)} \leq \|\overline{F}^{r,h,\tau}\|_{L^\infty(L^2)} = \max_{0 \leq m \leq m_0} \|F^m\|_{L^2} \leq C, \tag{4.30}
\]
\[
\sum_{m=1}^{m_0} \|F^m - F^{m-1}\|_{L^2}^2 = \tau \sum_{m=1}^{m_0} \|d_t F^m\|_{L^2}^2 \leq C. \tag{4.31}
\]

Then, based upon (4.20), (4.22), (4.30) and (4.31), there exists a convergent subsequence of \( \{\overline{F}^{r,h,\tau}\} \) (denoted by the same notation) \([10, 25]\) and a function \( f^\tau \in L^\infty((0, T_0); BV(\Omega)) \cap H^1((0, T_0); L^2(\Omega)) \) such that as \( h, \tau \to 0 \)
\[
\overline{F}^{r,h,\tau} \rightharpoonup f^\tau \quad \text{weakly * in } L^\infty((0, T_0); L^2(\Omega)),
\]
\[
\text{weakly in } L^2((0, T_0); L^2(\Omega)),
\]
\[
\text{strongly in } L^p(\Omega), 1 \leq p < \frac{n}{n-1}, \quad \text{for a.e. } t \in [0, T_0],
\]
and
\[
\overline{F}^{r,h,\tau} \rightarrow f_t^\tau \quad \text{weakly in } L^2((0, T_0); L^2(\Omega)). \tag{4.33}
\]
In (4.32), the fact that \(BV(\Omega)\) is compactly embedded in \(L^p(\Omega)\) for \(1 \leq p < \frac{n}{n-1}\) is used. Using the assumption on \(F^0\), we have \(f'(0) = f_0\). As authors have done in [10, 16], we obtain that \(f' \in L^\infty((0, T_0); BV(\Omega)) \cap H^1((0, T_0); L^2(\Omega))\) satisfies the following inequality formulation

\[
\int_0^s (f', (v - f')) dt + \int_0^s (E_{\lambda, e}(v) - E_{\lambda, e}(f')) dt \geq 0,
\]

where \(\forall s \in [0, T_0]\) and \(\forall v \in L^1((0, T_0); BV(\Omega)) \cap L^2(\Omega_{T_0})\). In addition, if we have two functions \(f'_0 \in L^\infty((0, T_0); BV(\Omega)) \cap H^1((0, T_0); L^2(\Omega)), i = 1, 2\) both satisfying (4.34) for given initial conditions \(f'_0(0)\) and observed data \(g_i\), respectively. Then, we have

\[
\|f'_1(s) - f'_2(s)\|_{L^2} \leq \|f'_1(0) - f'_2(0)\|_{L^2} + \|g_1 - g_2\|_{L^2}, \quad \forall s \in [0, T_0].
\]

Hence from (4.35), it is easy to see that the convergent solution \(f'\) of semi-implicit finite element discretization for (3.10) is uniqueness for given initial condition \(f_0\) and observed data \(g\). Next we further obtain that the whole sequence \(\{\tilde{F}^{\epsilon,h,\tau}\}\) converges to \(f'\). Therefore, the proof of (4.4) is completed.

In addition, it is easy to show that

\[
\|\tilde{F}^{\epsilon,h,\tau} - F^{\epsilon,h,\tau}\|_{L^\infty(L^2)}^2 = \max_{0 \leq \ell \leq T_0} \|\tilde{F}^{\epsilon,h,\tau} - F^{\epsilon,h,\tau}\|_{L^2}^2
\]

\[
= \max_{1 \leq m \leq m_0} \max_{m_{l-1} \leq \ell \leq m_t} \|\tilde{F}^{\epsilon,h,\tau} - F^{\epsilon,h,\tau}\|_{L^2}^2
\]

\[
= \max_{1 \leq m \leq m_0} \max_{m_{l-1} \leq \ell \leq m_t} \left(\frac{l - l_m}{\tau}\right)^2 \|F^m - F^{m-1}\|_{L^2}^2
\]

\[
\leq \max_{1 \leq m \leq m_0} \|F^m - F^{m-1}\|_{L^2}^2
\]

\[
\leq \sum_{m=1}^{m_0} \tau^2 \|d_t F^m\|_{L^2}^2
\]

According to (4.20), we have

\[
\|\tilde{F}^{\epsilon,h,\tau} - F^{\epsilon,h,\tau}\|_{L^\infty(L^2)} \leq C\tau^2.
\]

Then for \(1 < p < \frac{n}{n-1}\),

\[
\|\tilde{F}^{\epsilon,h,\tau} - F^{\epsilon,h,\tau}\|_{L^\infty(L^p)} \leq \tilde{C}\|\tilde{F}^{\epsilon,h,\tau} - F^{\epsilon,h,\tau}\|_{L^\infty(L^2)} \leq C\tau^2,
\]

where \(\tilde{C}\) is a positive constant. Using (4.4) and (4.36), we obtain (4.5). Therefore, for our semi-implicit finite element scheme (3.10) the proof of convergence is completed.

**Remark 4.1.** If the solution \(f' \in L^\infty((0, T_0); W^{1,1}(\Omega)) \cap L^\infty((0, T_0); H^1_{loc}(\Omega))\) of gradient flow (3.7) is existence and uniqueness for given initial value \(f'_0 = f_0\), then we have \(\tilde{f}' = f'\). However, if the solution is not exit, we just obtain a result that the semi-implicit finite element discretization is convergent.

**Theorem 4.2.** Suppose that the sequence \(\{E_{\lambda, e}(F^m)\}_{m=0}^\infty\) is obtained from the iterative scheme (3.10). The following result is immediately valid,

\[
E_{\lambda, e}(F^m) < E_{\lambda, e}(F^{m-1}), \quad m = 1, 2, \cdots, \infty,
\]

(4.37)
namely, the energy functional is strictly decrease. Moreover, we have

\[ \lim_{m \to \infty} F^m = f^*, \tag{4.38} \]

where the \( f^* \in \text{BV}(\Omega) \).

Proof. Now we again present the inequality (4.19)

\[
\|d_t F^m\|_{L^2}^2 + \frac{d_t \|PF^m - g\|_{L^2}^2}{2} + \tau \frac{\|d_t (PF^m - g)\|_{L^2}^2}{2} + \lambda d_t \int_{\Omega} \phi(\|\nabla F^m\|) \, dx \leq 0. 
\]

According to the definition of \( d_t \cdot \), we have

\[
E_{\lambda, \epsilon}(F^m) - E_{\lambda, \epsilon}(F^{m-1}) \leq -\tau \|d_t F^m\|_{L^2}^2 - \frac{\tau^2}{2} \|d_t (PF^m - g)\|_{L^2}^2 < 0, 
\]

hence the inequality (4.37) is verified and the sequence \( \{F^m\}_{m=0}^\infty \) is a minimizing sequence for the minimization problem (3.3). As a result of Theorem 2.1, since energy functional \( E_{\lambda, \epsilon}(f) \) is strictly convex and coercive, so a unique \( f^* \in \text{BV}(\Omega) \) is exist, such that

\[ \lim_{m \to \infty} F^m = f^*. \]

Therefore, this completes the proof of Theorem 4.2.

Combining Theorem 4.1 and Theorem 4.2, we immediately conclude that our semi-implicit finite element method is convergence, namely, the semi-implicit finite element solution approximates to the solution of the \( L^2 \)-gradient flow equation. More importantly, when the time goes to infinity, the semi-implicit finite element solution can achieve its steady state which approximates to the solution of corresponding Euler-Lagrange equations.

5. Numerical Experiments

In this section, we present several numerical experiments using synthetic and real data to illustrate that our semi-implicit finite element method can produce desirable image reconstruction results especially for the data detected from uniformly but very few or randomly distributed views and contaminated by random noise. We first suppose that the projections from one view are uniformly distributed. Actually, this hypothesis is reasonable for real tomography.

We take \( \phi \) as (3.8) which is a strictly convex function. The parameter \( \epsilon \) involved in \( \phi \) is a small number that leads to the perturbation of TV functional. In our numerical experiments we choose \( \epsilon = 0.001 \) and the temporal step size \( \tau = 0.01 \). The choice of factor \( \lambda \), however, depends only on the detected data that affects the reconstruction quality. If the \( \lambda \) is too small or too large, the reconstructed image contains considerable stripe artifacts and noise or smoothed edges which is demonstrated in our numerical simulations, respectively. When the additive Gaussian white noise is added to the projections, we define the projection SNR in decibels to be

\[
\text{SNR} = 10 \log \frac{1}{V P \sigma^2} \sum_{\nu=0}^{V-1} \sum_{p=0}^{P-1} |g(\theta_{\nu}, y_p)|^2, \tag{5.1}
\]

where \( V \) is the number of the angles, \( P \) is the number of projections from each angle and \( \sigma^2 \) is the variance of noise [5].
An originally synthetic Shepp-Logan phantom on the grid $512 \times 512$ used in the simulations is shown in Fig. 5.1, with the gray level in $[0.0, 1.0]$. We first consider the performance of our method resistant to projection data detected from a uniformly sparse angles with random noise. For this simulation, we obtain uniformly spacing 729 projections from each sampled view. The total angles is 60, namely, projecting once every 3 degrees. Then the additive Gaussian white noise is added to the projections resulting in a set of data SNR of 20dB. Image reconstruction for uniformly sparse projections (60 views uniformly distributed over $[0, \pi]$) contaminated by random noise (SNR = 20dB) is shown in Fig. 5.1. As shown in Fig. 5.1, after 40 iterations the quality of reconstructed image by our method is much better than that of FBP. As the increase of $\lambda$ in the finite interval, the stripe artifacts and noise are significantly reduced. If the $\lambda$, however, is too large, the edges of the reconstructed image are smeared. On the other hand,

\begin{figure}
\begin{center}
\includegraphics[width=0.8\textwidth]{image.png}
\end{center}
\caption{The SNR = 20dB. Top row: Reconstructed images by FBP (left) and our method with $\lambda = 2.0$ (middle), $\lambda = 3.0$ (right). Bottom row: Reconstructed images by our method with $\lambda = 4.0$ (left) and $\lambda = 5.0$ (middle), respectively. The original image is displayed on the $512 \times 512$ grid (right) with the gray level over $[0.0, 1.0]$.}
\end{figure}

we give the quantity of image error for comparison in Tab. 5.1 besides illustration by figures. In what follows we compute the image error

\begin{equation}
\text{err} = \frac{\|F - f\|_p}{\|f\|_p},
\end{equation}

where $F$ is the reconstructed image, $f$ is the original image and $\| \cdot \|_p$ denotes the $L^p$ norm of the image.

\begin{table}
\caption{The image errors in Fig. 5.1}
\end{table}
Another real data is a cross section of bacterial chaperonin GroEL (1J4Z) obtained from Protein Data Bank (PDB) which is displayed on a 120 × 120 grid as shown in Fig. 5.2. The sampled projection angles are the same as that of synthetic Shepp-Logan phantom. And we obtain uniformly spacing 171 projections from each sampled view. After 40 iterations, the reconstructive results by our method are illustrated. The projection SNR = 35dB in Fig. 5.2 and SNR = 25dB in Fig. 5.3. The corresponding image errors are evaluated in Tab. 5.2 and Tab. 5.3, respectively.

![Image of reconstructed images](image-url)

**Fig. 5.2.** The SNR = 35dB. Top row: Reconstructed images by FBP (left) and our method with $\lambda = 0.015$ (middle), $\lambda = 0.025$ (right). Bottom row: Reconstructed images by our method with $\lambda = 0.035$ (left) and $\lambda = 0.045$ (middle), respectively. The original image is displayed on the 120 × 120 grid (right) with the gray level over [-0.15, 1.21].
Fig. 5.3. The SNR = 25dB. Top row: Reconstructed images by FBP (left) and our method with $\lambda = 0.1$ (middle), $\lambda = 0.15$ (right). Bottom row: Reconstructed images by our method with $\lambda = 0.2$ (left), $\lambda = 0.25$ (middle), $\lambda = 0.3$ (right), respectively. The original image is displayed on the 120 x 120 grid (right) with the gray level over [-0.15, 1.21] as shown in Fig. 5.2.

Comparing the two figures, we can find the fact that when the SNR becomes smaller, the parameter $\lambda$ becomes larger to remove the stripe artifacts and noise due to sparse projection angles. As shown in Fig. 5.1-5.3, it is a fact that the performance of our method far exceeds that of FBP when the set of projection data detected from uniformly sparse angles is contaminated by random noise.

In addition, in the simulation experiment, Fig. 5.4 shows the random-angle FBP reconstruction and the reconstructed images by our method after 40 iterations from noise projections (SNR = 25dB) where the set of detected data is projected from 180 random angles and uniformly spacing 729 projections from each sampled view. The corresponding evaluation of image errors is presented in Tab. 5.4.

<table>
<thead>
<tr>
<th>err</th>
<th>FBP</th>
<th>$\lambda = 0.1$</th>
<th>$\lambda = 0.15$</th>
<th>$\lambda = 0.2$</th>
<th>$\lambda = 0.25$</th>
<th>$\lambda = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 1$</td>
<td>0.058684</td>
<td>0.026433</td>
<td>0.025374</td>
<td>0.025104</td>
<td>0.025211</td>
<td>0.025513</td>
</tr>
<tr>
<td>$p = 2$</td>
<td>0.074877</td>
<td>0.037294</td>
<td>0.036612</td>
<td>0.036746</td>
<td>0.037280</td>
<td>0.038011</td>
</tr>
<tr>
<td>$p = \infty$</td>
<td>0.366064</td>
<td>0.248988</td>
<td>0.231261</td>
<td>0.213586</td>
<td>0.193321</td>
<td>0.197353</td>
</tr>
</tbody>
</table>
Fig. 5.4. The SNR = 25dB. Top row: Reconstructed images by FBP (left) and our method with \( \lambda = 2.0 \) (middle), \( \lambda = 3.0 \) (right). Bottom row: Reconstructed images by our method with \( \lambda = 4.0 \) (left), \( \lambda = 5.0 \) (middle), \( \lambda = 6.0 \) (right), respectively. The original image is displayed on the 512 × 512 grid with the gray level over [0.0, 1.0] as shown in Fig. 5.1.

In the real data experiment, Fig. 5.5 shows the results of the random-angle FBP reconstruction and the reconstructions by our method after 40 iterations from noise projections (SNR = 25dB) where the set of detected data is projected from 180 random angles and uniformly spacing 171 projections from each sampled view. As shown in Fig. 5.5, it is a fact that the performance of our method far exceeds that of FBP when the set of projection data detected from random angles is contaminated by random noise. The corresponding evaluation of image errors is presented in Tab. 5.5.

Tab. 5.4: The image errors in Fig. 5.4

<table>
<thead>
<tr>
<th>err</th>
<th>FBP</th>
<th>( \lambda = 2.0 )</th>
<th>( \lambda = 3.0 )</th>
<th>( \lambda = 4.0 )</th>
<th>( \lambda = 5.0 )</th>
<th>( \lambda = 6.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 1 )</td>
<td>0.131736</td>
<td>0.021923</td>
<td>0.019250</td>
<td>0.018025</td>
<td>0.017736</td>
<td>0.017968</td>
</tr>
<tr>
<td>( p = 2 )</td>
<td>0.184256</td>
<td>0.047897</td>
<td>0.046746</td>
<td>0.047635</td>
<td>0.049464</td>
<td>0.051748</td>
</tr>
<tr>
<td>( p = \infty )</td>
<td>1.194048</td>
<td>0.603087</td>
<td>0.628351</td>
<td>0.647875</td>
<td>0.660598</td>
<td>0.668552</td>
</tr>
</tbody>
</table>

Tab. 5.5: The image errors in Fig. 5.5

<table>
<thead>
<tr>
<th>err</th>
<th>FBP</th>
<th>( \lambda = 0.125 )</th>
<th>( \lambda = 0.15 )</th>
<th>( \lambda = 0.175 )</th>
<th>( \lambda = 0.2 )</th>
<th>( \lambda = 0.225 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 1 )</td>
<td>0.139320</td>
<td>0.023148</td>
<td>0.023054</td>
<td>0.023081</td>
<td>0.023192</td>
<td>0.023364</td>
</tr>
<tr>
<td>( p = 2 )</td>
<td>0.199221</td>
<td>0.033293</td>
<td>0.033339</td>
<td>0.033524</td>
<td>0.033805</td>
<td>0.034155</td>
</tr>
<tr>
<td>( p = \infty )</td>
<td>0.982599</td>
<td>0.280780</td>
<td>0.258409</td>
<td>0.236512</td>
<td>0.215437</td>
<td>0.198456</td>
</tr>
</tbody>
</table>
From Fig. 5.4-5.5, it is easy to see that the FBP method obtained bad results for noise random-angle tomographic reconstruction. Our algorithm, however, yields desirable reconstructed images. These conclusions are also verified by the evaluation of image errors, see Tab. 5.4-5.5.

6. Conclusion and Future Work

We have presented a novel and effective $L^2$-gradient flow based semi-implicit finite element method for solving the variational model of image reconstruction under various data scenarios, especially for the contaminated data detected from uniformly few or randomly distributed projection angles. In addition, we have given a rigorous proof for the convergence of the semi-implicit finite element method. Finally, various numerical experiments are presented which demonstrate that our method is stable and effective. The performance of tomographic reconstruction is desirable which far exceeds that of FBP. Moreover, our approach behaves good especially for random projection data with noise. Hence, our new algorithm can be extended to cryo-ET and cryo-EM reconstruction. Therefore, the future work involves generalizing the model to high dimensions, particularly, to the research field of single particle cryo-EM reconstruction.
References


