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by

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An iterative method for computing Beltrami fields on bounded domains *

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Abstract

In this paper, we are concerned with numerical methods for solving the Beltrami field equations. By introducing a new variable, we transform the original nonlinear problem into a minimization problem involving two unknowns. Then we propose an iterative method for solving the minimization problem so that the two unknowns can be solved respectively in each iterative step. The convergence of the new iterative method is proved. Moreover, it is shown that the resulting approximate solution possesses the optimal error estimate in some sense. The theoretical results are confirmed by numerical experiments.

Key Words: Beltrami fields, non-linear PDE, optimization procedure, convergence, iterative method, finite element methods, error estimate.

1 Introduction

We consider the following Beltrami field equations

$$\mathbf{curl}\ \mathbf{u} \times \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1}$$

and

$$\text{div}\ \mathbf{u} = 0 \quad \text{in } \Omega. \tag{2}$$

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Here $\mathbf{u}$ is the magnetic field vector and Ω is an open subset of $\mathbb{R}^3$ with boundary $\partial\Omega$. The domain Ω is assumed bounded, simply connected, and its boundary $\partial\Omega$ is either Lipschiz-continuous or $\mathcal{C}^{1,1}$ continuous. In the present paper, we only consider the case with $\partial\Omega$ being connected, while our results can be extended to other situations (which will be discussed in another paper). Beltrami fields are particularly a subject of an intensive research in solar physics [6], [7], [21]. The equations (1) and (2) are often encountered in the magnetic models of outer atmospheres above the low solar corona [1]. It is difficult to design an efficient method for solving the Beltrami field equations because the equations are nonlinear and have many solutions.

In resent years, how to solve the equations (1) and (2) has attracted wide attention of many researchers. Various numerical methods were proposed, for example, optimization scheme, Grad-Rubin method, boundary integral method and MHD evolutionary technique, etc. For the readers' convenience, we recall the basic ideas of the optimization scheme and the Grad-Rubin method in the following.

The optimization scheme for solving the Beltrami field equations was first proposed in [23], and later improved in [21] and [24]. The basic idea of the method is to minimize an objective functional L containing the normalized magnetic force and the divergence-free condition:

$$\min_{\forall \mathbf{u}} L(\mathbf{u}) = \frac{1}{|\Omega|}\int_\Omega \Big(\frac{|\mathbf{curl\ u} \times \mathbf{u}|^2}{|\mathbf{u}|^2} + |\mathrm{div}\ \mathbf{u}|^2\Big)\ d\mathbf{x}. \tag{3}$$

Since the functional L is not convex on $\mathbf{u}$, the minimization problem is difficult to solve, so an artificial time variable t has to be introduced into $\mathbf{u}$ (see [23] for the details). Furthermore, the minimization problem (3) can be transformed into a partial differential equation which contains a derivative of $\mathbf{u}$ with respect to t. By the forward Euler discretization for the time variable t, an explicit iterative framework is designed to update the magnetic field vector $\mathbf{u}$. In the strategy a very small time step is required to ensure numerical stability, which makes the method quite slow for large scale problems.

According to the equation (1), it is easy to know that the filed $\mathbf{u}$ and its current density $\mathbf{curl\ u}$ are parallel. Hence, the equation (1) is equivalent to the following equation

$$\mathbf{curl\ u} = \lambda \mathbf{u} \quad \text{in } \Omega. \tag{4}$$

The magnetic field vector $\mathbf{u}$ is called a nonlinear force-free field when the function λ is not constant everywhere. The Grad-Rubin method, which was first proposed in [13], is a current-field iteration for solving some equations governing the equilibrium of a perfectly conducting fluid in the presence of a magnetic field. This method was developed to reconstruct nonlinear force-free fields in [2], [16] and [22]. The basic idea of the Grad-Rubin scheme is to transform the equations (4) and (2) into two systems: a hyperbolic system corresponding to the transport of λ along field lines, and an elliptic system updating the magnetic field configuration $\mathbf{u}$. An important feature of the Grad-Rubin scheme is that one only needs to solve two linear systems with the unknowns $\mathbf{u}$ and λ respectively instead of the original nonlinear equation (1). Moreover, the two unknowns $\mathbf{u}$ and λ can be solved respectively in each iterative step. Although Grad-Rubin method has been studied for many years, there is no any theoretical analysis on the convergence of this method. Furthermore, as pointed out in [2], the convergence of this method loses for a large boundary data on λ.

Although these two strategies introduced above have been applied widely to reconstruct the Beltrami fields, neither theoretical nor numerical reliability of them can be guaranteed. In fact, the convergence of almost all existing methods for computing Beltrami fields has not been shown yet. The purpose of the present paper is to propose a reliable iterative method for solving the Beltrami field equations.

In this paper, we consider the following minimization problem

$$\min_{\forall \mathbf{u},\lambda} E(\mathbf{u},\lambda) = \frac{1}{2}\int_\Omega |\mathbf{curl}\ \mathbf{u} - \lambda\mathbf{u}|^2\ d\mathbf{x}, \quad \text{subject to}\ \ \text{div}\ \mathbf{u} = 0 \text{ in } \Omega. \tag{5}$$

It is clear that the solutions of the equations (4) and (2) are also the solutions of the minimization problem (5). Note that the objective functional E is a convex quadratic functional on $\mathbf{u}$ and λ respectively. It means that when λ (rep. $\mathbf{u}$) is given, the functional E is a convex quadratic functional only depending on $\mathbf{u}$ (rep. λ), which is our motivation to choose E as the objective functional. Motivated by the idea of the Grad-Rubin method, we transform the minimization problem (5) into two different minimization problems so that $\mathbf{u}$ and λ can be solved respectively and alternately. Based on this framework, we propose an iterative method for solving the minimization problem (5). The new iterative method has obvious advantages over the Grad-Rubin scheme: (i) in the step to update λ, only a quadratic programming problem needs to be solved by the conjugate gradient method, while a troublesome hyperbolic problem on λ has to be solved in the Grad-Rubin scheme; (ii) the convergence of the proposed method can be strictly proved; (iii) our numerical results indicate that the proposed method is still convergent even if the boundary data of λ is large.

The paper is organized as follows. In section 2, we introduce the basic idea and derive other two minimization problems from the minimization problem (5). In section 3 and section 4, we investigate some properties of the two minimization problems introduced in Section 2 respectively. In section 5, we propose an iterative method and prove its convergence. In section 6, we give an error estimate of the resulting approximate solution. In section 7, some numerical results are reported to confirm the theoretical results.

2 Preliminaries

In this section, we recall the Beltrami field problem and introduce our basic idea for designing a new iterative method to solve the Beltrami field equations. First of all, we give the mathematical statement of the origin problem.

2.1 The Beltrami field problem

Throughout this paper, the usual Sobolev spaces are denoted by the standard notations, and the norm on the Sobolev space V is denoted by $\|\cdot\|_V$. For example, the dual space of $H^{1/2}(\partial\Omega)$ is denoted by $H^{-1/2}(\partial\Omega)$, and the norm on $H^{-1/2}(\partial\Omega)$ is denoted by $\|\cdot\|_{H^{-1/2}(\partial\Omega)}$. In addition, the Sobolev space $H(\mathbf{curl};\Omega)$ (resp. $H(\text{div};\Omega)$) indicates the set of the vector functions $\mathbf{v}\in L^2(\Omega)^3$ such that $\mathbf{curl}\ \mathbf{v}\in L^2(\Omega)^3$ (resp. div $\mathbf{v}\in L^2(\Omega)$). Let $H_0(\mathbf{curl};\Omega)$ be the space

$$H_0(\mathbf{curl};\Omega)=\{\mathbf{v}\in H(\mathbf{curl};\Omega);\mathbf{v}\times\mathbf{n}|_{\partial\Omega}=0\}.$$

Also, we introduce the Sobolev space $\mathcal{X}_{\partial\Omega}$ defined as

$$\mathcal{X}_{\partial\Omega}=\{\mathbf{v}\in H^{-1/2}(\partial\Omega)^3;\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0 \text{ and } \text{div}_\tau\mathbf{v}\in H^{-1/2}(\partial\Omega)\},$$

endowed with the norm

$$\|\mathbf{v}\|_{\mathcal{X}_{\partial\Omega}}=\|\mathbf{v}\|_{H^{-1/2}(\partial\Omega)^3}+\|\text{div}_\tau\mathbf{v}\|_{H^{-1/2}(\partial\Omega)},$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and $\text{div}_\tau\mathbf{v}$ is the tangential divergence of $\mathbf{v}$ on $\partial\Omega$ (refer to [4]).

Let $\mathbf{u}\in H(\mathbf{curl};\Omega)\cap H(\text{div};\Omega)$ and $\lambda\in W^{1,\infty}(\bar{\Omega})$ satisfy

$$\begin{cases}\mathbf{curl}\ \mathbf{u}=\lambda\mathbf{u} & \text{in } \Omega,\\ \text{div}\ \mathbf{u}=0 & \text{in } \Omega.\end{cases}\tag{6}$$

In the following, we investigate how to give the boundary conditions of $\mathbf{u}$ and λ.

Denote the traces of $\mathbf{u}$ and λ on $\partial\Omega$ by $\mathbf{u}_0$ and λ_0 respectively. Supposing that $\tilde{\mathbf{f}}=\lambda\mathbf{u}$ is known, we consider the following problem : Find $\mathbf{u}\in H(\mathbf{curl};\Omega)\cap H(\text{div};\Omega)$ such that

$$\begin{cases}\mathbf{curl}\ \mathbf{u}=\tilde{\mathbf{f}} & \text{in } \Omega,\\ \text{div}\ \mathbf{u}=0 & \text{in } \Omega.\end{cases}\tag{7}$$

It is well known that the magnetic field $\mathbf{u}_0$ is observable. Although the three components of $\mathbf{u}_0$ are known, only a part of this information can be imposed on the boundary of Ω in order to get a well posed problem. Following the discussion in [4], the boundary condition for the problem (7) can be prescribed as

$$\mathbf{u} \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} \quad \text{on } \partial\Omega. \tag{8}$$

For the system (6), we also use (8) to prescribe the boundary condition of $\mathbf{u}$.

In the same way, it is natural to give the boundary condition of λ by

$$\lambda = \lambda_0 \text{ on } \partial\Omega. \tag{9}$$

In order to guarantee the mathematical statement of the origin problem being well defined (see [4]), $\mathbf{u}_0$ and λ_0 must satisfy the constraint

$$\mathrm{div}_\tau(\mathbf{u}_0 \times \mathbf{n}) = \lambda_0(\mathbf{u}_0 \cdot \mathbf{n}) \text{ on } \partial\Omega. \tag{10}$$

This implies that, for each point $\mathbf{x} \in \partial\Omega$, if $\mathbf{u}_0 \cdot \mathbf{n}(\mathbf{x}) = 0$, then $\mathrm{div}_\tau(\mathbf{u}_0 \times \mathbf{n})(\mathbf{x})$ must vanish, and λ_0 does not contribute to the equality (10).

According to the discussion above, the mathematical statement of the Beltrami field problem is: Find $\mathbf{u}^\dagger \in H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$ and $\lambda^\dagger \in W^{1,\infty}(\bar{\Omega})$ such that

$$\begin{cases} \mathbf{curl}\ \mathbf{u}^\dagger = \lambda^\dagger \mathbf{u}^\dagger & \text{in } \Omega, \\ \mathrm{div}\ \mathbf{u}^\dagger = 0 & \text{in } \Omega, \\ \mathbf{u}^\dagger \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} & \text{on } \partial\Omega, \\ \lambda^\dagger = \lambda_0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where the two known functions $\mathbf{u}_0$ and λ_0 satisfy (10). Analogous mathematical statements of the nonlinear force-free field problem can be found in [5], [6] and [8].

Remark 1. *From (10), we know* $\mathrm{div}_\tau(\mathbf{u}_0 \times \mathbf{n}) = 0$ *on* $\partial\Omega$ *when* $\lambda_0 = 0$ *on* $\partial\Omega$*. Furthermore, when* $\lambda_0 = 0$ *on* $\partial\Omega$*, it follows from Theorem 4.1 in [4] that the problem (11) admits one solution* $(\mathbf{u}^\dagger, \lambda^\dagger) = (\mathbf{u}^\dagger, 0)$*. In the case with* $\lambda_0 = 0$ *on* $\partial\Omega$*, we only need to solve (11) with* $\lambda^\dagger = 0$ *. Since the equations (11) with* $\lambda^\dagger = 0$ *are the Maxwell equations, we can solve them by the conventional ways.*

In the rest of this paper, we assume that the problem (11) always possesses at least one solution, and only consider the case in which there is a nonempty open subset Σ of $\partial\Omega$ such that $\lambda_0 \neq 0$ on Σ.

2.2 The Motivation

As pointed out in Section 1, we want to design an iterative framework for solving (11), where the two variables $\mathbf{u}$ and λ can be updated respectively in two iteration steps.

After λ is obtained through one iterative step, we try to update $\mathbf{u}$ by solving the following equations in the other iteration step,

$$\begin{cases} \mathbf{curl}\ \mathbf{u} - \lambda\mathbf{u} = 0 & \text{in } \Omega, \\ \text{div }\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}\times\mathbf{n} = \mathbf{u}_0\times\mathbf{n} & \text{on } \partial\Omega. \end{cases} \tag{12}$$

However, we can not guarantee that the equations (12) with a given function λ always admit at least one solution, because the function λ changes in the iteration process and may not be the solution $\lambda^\dagger$ of (11). Therefore, we have to give up the standard idea mentioned above, and would like to transform the problem (12) into the following least squares problem,

$$\min_{\forall\mathbf{u}} \frac{1}{2}\|\mathbf{curl}\ \mathbf{u} - \lambda\mathbf{u}\|^2_{L^2(\Omega)^3}, \tag{13}$$

subject to

$$\begin{cases} \text{div }\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}\times\mathbf{n} = \mathbf{u}_0\times\mathbf{n} & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Namely, in each iteration step for computing $\mathbf{u}$ with a known λ, we solve the minimization problem (13) instead of solving (12) directly. As we will see, the minimization problem (13) always admits one solution. Hence, in order to construct the desired iterative method, we consider the following minimization problem on $\mathbf{u}^\dagger \in H(\mathbf{curl};\Omega)$ and $\lambda^\dagger \in W^{1,\infty}(\bar{\Omega})$,

$$\min_{\forall\mathbf{u}^\dagger,\lambda^\dagger} E(\mathbf{u}^\dagger,\lambda^\dagger) = \frac{1}{2}\|\mathbf{curl}\ \mathbf{u}^\dagger - \lambda^\dagger\mathbf{u}^\dagger\|^2_{L^2(\Omega)^3}, \tag{15}$$

subject to

$$\begin{cases} \text{div }\mathbf{u}^\dagger = 0 & \text{in } \Omega, \\ \mathbf{u}^\dagger\times\mathbf{n} = \mathbf{u}_0\times\mathbf{n} & \text{on } \partial\Omega, \\ \lambda^\dagger = \lambda_0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Under the assumption that the problem (11) has at least one solution, it is obvious that the problem (11) is equivalent to the minimization problem (15) with the constraint (16).

According to the discussion above, we only need to design a suitable iterative method for solving the minimization problem (15) instead of the problem (11) itself. In the desired iterative method, we expect to solve two subproblems alternatively. One subproblem is to find $\mathbf{u}^* \in H(\mathbf{curl};\Omega)$ such that

$$\mathbf{u}^* := \arg\min_{\forall\mathbf{u}\in H(\mathbf{curl};\Omega)} \frac{1}{2}\|\mathbf{curl}\ \mathbf{u} - \lambda\mathbf{u}\|^2_{L^2(\Omega)^3}, \tag{17}$$

subject to

$$\text{div }\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u}\times\mathbf{n} = \mathbf{u}_0\times\mathbf{n} \text{ on } \partial\Omega,$$

where $\lambda \in W^{1,\infty}(\bar{\Omega})$ is a known function. The other is to find $\lambda^* \in W^{1,\infty}(\bar{\Omega})$ such that

$$\lambda^* := \arg \min_{\forall \lambda \in W^{1,\infty}(\bar{\Omega})} \frac{1}{2} \|\mathbf{curl}\ \mathbf{u} - \lambda \mathbf{u}\|^2_{L^2(\Omega)^3}, \text{ subject to } \lambda = \lambda_0 \text{ on } \partial\Omega, \tag{18}$$

where $\mathbf{u}$ is a known function belonging to $H(\mathbf{curl};\Omega)$. Before describing the new iterative method, we introduce numerical methods for solving the problems (17) and (18) in the next two sections respectively.

3 A solution method of the optimization problem (17)

In this section, we consider the minimization problem on $\mathbf{u}^*$ with a known function $\lambda \in W^{1,\infty}(\bar{\Omega})$,

$$\min_{\forall \mathbf{u}^* \in H(\mathbf{curl};\Omega)} E(\mathbf{u}^*, \lambda) = \frac{1}{2} \|\mathbf{curl}\ \mathbf{u}^* - \lambda \mathbf{u}^*\|^2_{L^2(\Omega)^3}, \tag{19}$$

subject to

$$\text{div } \mathbf{u}^* = 0 \text{ in } \Omega, \quad \mathbf{u}^* \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} \text{ on } \partial\Omega.$$

The numerical analysis in this section is based on the assumption that either $\lambda = 0$ in $\bar{\Omega}$ or $\lambda \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$. As we will see, the assumption is satisfied in the step to update λ.

Find a divergence-free function $\tilde{\mathbf{u}}_0 \in H(\mathbf{curl};\Omega)$ such that $\tilde{\mathbf{u}}_0 \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$. Let $\mathbf{u} = \mathbf{u}^* - \tilde{\mathbf{u}}_0$ and $\mathbf{f}_\lambda = \lambda \tilde{\mathbf{u}}_0 - \mathbf{curl}\ \tilde{\mathbf{u}}_0$. The problem (19) can be written as

$$\min_{\forall \mathbf{u} \in H_0(\mathbf{curl};\Omega)} E(\mathbf{u}, \lambda) = \frac{1}{2} \|\mathbf{curl}\ \mathbf{u} - \lambda \mathbf{u} - \mathbf{f}_\lambda\|^2_{L^2(\Omega)^3}, \tag{20}$$

subject to

$$\text{div } \mathbf{u} = 0 \text{ in } \Omega. \tag{21}$$

Let

$$\mathcal{D} = \{\psi \in H^1(\Omega) \cap L^2_0(\Omega);\ \psi|_{\partial\Omega} = \alpha,\ \alpha \in \mathbb{R}\},$$

where $L^2_0(\Omega) = L^2(\Omega)/\mathbb{R}$. Define the bilinear forms

$$a_\lambda(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{curl}\ \mathbf{u} - \lambda \mathbf{u}) \cdot (\mathbf{curl}\ \mathbf{v} - \lambda \mathbf{v})\ d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in H(\mathbf{curl};\Omega),$$

and

$$b(\mathbf{v}, \psi) = \int_\Omega \mathbf{v} \cdot \nabla\psi\ d\mathbf{x}, \quad \forall \mathbf{v} \in H(\mathbf{curl};\Omega),\ \forall \psi \in \mathcal{D},$$

and the linear form

$$(\mathbf{F}_\lambda, \mathbf{v}) = \int_\Omega \mathbf{f}_\lambda \cdot (\mathbf{curl}\ \mathbf{v} - \lambda \mathbf{v})\ d\mathbf{x}, \quad \forall \mathbf{v} \in H(\mathbf{curl};\Omega).$$

Following the standard technique (refer to [11]), the minimization problem (20) with the constraint (21) can be transformed into the saddle-point problem: Find $(\mathbf{u}, \varphi) \in H_0(\mathbf{curl}; \Omega) \times \mathcal{D}$ such that

$$\begin{cases} a_\lambda(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \varphi) = (\mathbf{F}_\lambda, \mathbf{v}), & \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega), \\ b(\mathbf{u}, \psi) = 0, & \forall \psi \in \mathcal{D}, \end{cases} \tag{22}$$

where φ is the Lagrange multiplier associated with the constraint (21).

3.1 On the saddle-point problem (22)

In this subsection, we prove that the saddle-point problem (22) admits one and only one solution, and check that the problem (20) is equivalent to the saddle-point problem (22).

Theorem 1. *Suppose that $\lambda \in W^{1,\infty}(\bar{\Omega})$ such that either $\lambda = 0$ in $\bar{\Omega}$ or $\lambda \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$. The problem (22) admits one and only one solution $(\mathbf{u}, \varphi) \in H_0(\mathbf{curl}; \Omega) \times \mathcal{D}$. Furthermore,* div $\mathbf{u} = 0$ *in Ω and*

$$\|\mathbf{u}\|_{H(\mathbf{curl};\Omega)} + \|\varphi\|_{H^1(\Omega)} \leqslant C\|\mathbf{f}_\lambda\|_{L^2(\Omega)^3}, \tag{23}$$

where C is a constant independent of $\mathbf{u}$ and φ.

□

Theorem 2. *Suppose that $\lambda \in W^{1,\infty}(\bar{\Omega})$ such that either $\lambda = 0$ in $\bar{\Omega}$ or $\lambda \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$. The problem (20) is equivalent to the saddle-point problem (22). Moreover, the minimization problem (19) is also equivalent to the saddle-point problem (22).*

□

Before presenting the proofs of the two theorems above, we need to prove a few auxiliary results.

The following result can be found in [12] and [19].

Lemma 1. *Let Ω_0 be a connected open subset of Ω and suppose $\mathbf{v} \in H^2(\Omega_0)^3$ where $\mathbf{v} = (v_1, v_2, v_3)^T$ satisfies*

$$|\Delta \mathbf{v}| \leqslant C \sum_{i=1}^{3} (|v_i| + |\nabla v_i|), \tag{24}$$

almost everywhere in Ω_0, where C is a constant. If $\mathbf{v}$ vanishes identically on a neighborhood of a point $\mathbf{x}_0 \in \Omega_0$, then $\mathbf{v}$ is identically zero in Ω_0.

□

Lemma 1 leads to the following result.

Lemma 2. *Suppose* Ω_1 *is an open connected subdomain of* Ω. *Assume that* $\mathbf{u} \in H(\mathbf{curl};\Omega_1) \cap H(\mathrm{div};\Omega_1)$ *satisfies*

$$\mathbf{curl}\ \mathbf{u} = \lambda \mathbf{u}, \quad \mathrm{div}\ \mathbf{u} = 0, \tag{25}$$

almost everywhere in Ω_1, *where* $\lambda \in W^{1,\infty}(\bar{\Omega}_1)$. *If* $\mathbf{u}$ *vanishes on a ball of non-zero radius contained in* Ω_1, *then* $\mathbf{u} = 0$ *in* Ω_1.

Proof. Since $\lambda \in W^{1,\infty}(\bar{\Omega}_1)$ and $\mathbf{curl}\ \mathbf{u} = \lambda \mathbf{u}$, we have

$$\mathbf{curl}\ \mathbf{curl}\ \mathbf{u} = \nabla\lambda \times \mathbf{u} + \lambda \mathbf{curl}\ \mathbf{u}, \tag{26}$$

and $\mathbf{curl}\ \mathbf{u} \in H(\mathbf{curl};\Omega_1)$. By the fact that $\mathbf{curl}\ \mathbf{curl}\ \mathbf{u} = -\Delta\mathbf{u} + \nabla(\mathrm{div}\ \mathbf{u})$ and $\mathrm{div}\ \mathbf{u} = 0$, we obtain

$$-\Delta\mathbf{u} = \nabla\lambda \times \mathbf{u} + \lambda \mathbf{curl}\ \mathbf{u}. \tag{27}$$

Hence $\Delta\mathbf{u} \in L^2(\Omega_1)^3$. From standard interior elliptic regularity results, we know that, for any subdomain Ω_2 which is compactly contained in Ω_1, $\mathbf{u} \in H^2(\Omega_2)^3$. By Lemma 1, we conclude $\mathbf{u} = 0$ in Ω_2. Note that the subdomain Ω_2 is arbitrary. Hence $\mathbf{u} = 0$ in Ω_1. □

Based on Lemma 2, we can prove the following result.

Lemma 3. *Assume that* $\mathbf{u} \in H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$ *satisfies*

$$\begin{cases} \mathbf{curl}\ \mathbf{u} = \lambda\mathbf{u} & \text{in } \Omega, \\ \mathrm{div}\ \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where $\lambda \in W^{1,\infty}(\bar{\Omega})$. *If* λ *satisfies that either* $\lambda = 0$ *in* $\bar{\Omega}$ *or* $\lambda \neq 0$ *on a boundary component* Σ *which is a nonempty open subset of* $\partial\Omega$, *then* $\mathbf{u} = 0$ *in* Ω.

Proof. When $\lambda = 0$ in $\bar{\Omega}$, we deduce that $\mathbf{u}$ satisfies

$$\mathbf{curl}\ \mathbf{u} = 0 \text{ in } \Omega, \quad \mathrm{div}\ \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{29}$$

Since $\partial\Omega$ is connected, it is obvious that $\mathbf{u} = 0$ in Ω. When $\lambda \neq 0$ on a boundary component Σ, we have $\mathbf{curl}\ \mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, because $\mathbf{u} \in H(\mathbf{curl};\Omega)$ and $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$. By the fact that $\mathbf{curl}\ \mathbf{u} = \lambda\mathbf{u}$ in Ω, we deduce $\lambda\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. In addition, since $\lambda \neq 0$ on Σ, we get $\mathbf{u} \cdot \mathbf{n} = 0$ on Σ. Let Ω_Σ be a subdomain in Ω such that $\bar{\Omega}_\Sigma \cap \Sigma$ contains a non-trivial open subset of Σ and $\lambda \in W^{1,\infty}(\bar{\Omega}_\Sigma)$. We extend λ to a function $\tilde{\lambda}$ defined on $\mathbb{R}^3$ such that $\tilde{\lambda} \in W^{1,\infty}(\mathbb{R}^3)$. Since Σ is an open subset of $\partial\Omega$, we can find an open ball $B(\mathbf{x}_0, r)$ of radius $r > 0$ centered at $\mathbf{x}_0$ on $\bar{\Omega}_\Sigma \cap \Sigma$ satisfying $B(\mathbf{x}_0, r) \cap \partial\Omega \subset \Sigma$ and $B(\mathbf{x}_0, r) \cap \Omega \subset \Omega_\Sigma$. Extending $\mathbf{u}$ by zero from Ω_Σ to $B(\mathbf{x}_0, r)\backslash\Omega_\Sigma$, we obtain

$$\begin{cases} \int_{B(\mathbf{x}_0,r)\cup\Omega_\Sigma} (\mathbf{curl}\ \mathbf{u} - \tilde{\lambda}\mathbf{u}) \cdot \mathbf{v}\ d\mathbf{x} = 0, & \forall \mathbf{v} \in H_0(\mathbf{curl}; B(\mathbf{x}_0, r) \cup \Omega_\Sigma), \\ \int_{B(\mathbf{x}_0,r)\cup\Omega_\Sigma} \mathrm{div}\ \mathbf{u} \cdot p\ d\mathbf{x} = 0, & \forall p \in H_0^1(B(\mathbf{x}_0, r) \cup \Omega_\Sigma). \end{cases} \tag{30}$$

Hence $\mathbf{u}$ satisfies (25) almost everywhere in $B(\mathbf{x}_0, r) \cup \Omega_\Sigma$. Furthermore, $\mathbf{u} = 0$ in $B(\mathbf{x}_0, r)\backslash\Omega_\Sigma$. It follows from Lemma 2 that $\mathbf{u} = 0$ in $B(\mathbf{x}_0, r) \cup \Omega_\Sigma$. Therefore, $\mathbf{u}$ vanishes on a ball of non-zero radius contained in $\Omega \cap B(\mathbf{x}_0, r)$. Now we show $\mathbf{u} = 0$ in Ω by Lemma 2 again. □

By Lemma 3, we can obtain an important inequality.

Lemma 4. *Assume that $\lambda \in W^{1,\infty}(\bar{\Omega})$ such that either $\lambda = 0$ in $\bar{\Omega}$ or $\lambda \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$. The following inequality is valid for every $\mathbf{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)$ satisfying $\mathrm{div}\ \mathbf{u} = 0$ in Ω,*

$$a_\lambda(\mathbf{u}, \mathbf{u}) \geqslant C_1\big(\|\mathbf{curl}\ \mathbf{u}\|^2_{L^2(\Omega)^3} + \|\lambda\mathbf{u}\|^2_{L^2(\Omega)^3}\big), \tag{31}$$

where C_1 is a constant independent of $\mathbf{u}$.

Proof. We prove the inequality (31) by the reductio ad absurdum. Assume the inequality above is false. Hence we can find a sequence $\{\mathbf{u}_n\}_{n=1}^\infty$ such that, for each positive integer n, $\mathbf{u}_n \in H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)$, $\mathrm{div}\ \mathbf{u}_n = 0$ and

$$\begin{cases} \|\mathbf{curl}\ \mathbf{u}_n\|^2_{L^2(\Omega)^3} + \|\lambda\mathbf{u}_n\|^2_{L^2(\Omega)^3} = 1, \\ a_\lambda(\mathbf{u}_n, \mathbf{u}_n) = \int_\Omega (\mathbf{curl}\ \mathbf{u}_n - \lambda\mathbf{u}_n) \cdot (\mathbf{curl}\ \mathbf{u}_n - \lambda\mathbf{u}_n)\ d\mathbf{x} < \frac{1}{n}. \end{cases} \tag{32}$$

Since $\mathbf{u}_n \in H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)$ such that $\mathrm{div}\ \mathbf{u}_n = 0$ and $\|\mathbf{curl}\ \mathbf{u}_n\|^2_{L^2(\Omega)^3} \leqslant 1$, we can extract a subsequence of $\{\mathbf{u}_n\}_{n=1}^\infty$ which converges weakly in $H^1(\Omega)^3$ and strongly in $L^2(\Omega)^3$. We still denote the subsequence by $\{\mathbf{u}_n\}_{n=1}^\infty$. Therefore, $\{\mathbf{u}_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\Omega)^3$. By the fact that $\|\mathbf{curl}\ \mathbf{u}_n - \lambda\mathbf{u}_n\|_{L^2(\Omega)^3} < 1/\sqrt{n}$ and $\mathrm{div}\ \mathbf{u}_n = 0$, we deduce, for any two positive integers m and n,

$$\begin{aligned} & \|\mathbf{curl}\ \mathbf{u}_m - \mathbf{curl}\ \mathbf{u}_n\|_{L^2(\Omega)^3} + \|\mathrm{div}\ \mathbf{u}_m - \mathrm{div}\ \mathbf{u}_n\|_{L^2(\Omega)} \\ \leqslant\ & \|\mathbf{curl}\ \mathbf{u}_m - \lambda\mathbf{u}_m\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{u}_n - \lambda\mathbf{u}_n\|_{L^2(\Omega)^3} \\ & + \|\lambda\|_{L^\infty(\Omega)}\|\mathbf{u}_m - \mathbf{u}_n\|_{L^2(\Omega)^3} \\ \leqslant\ & \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} + \|\lambda\|_{L^\infty(\Omega)}\|\mathbf{u}_m - \mathbf{u}_n\|_{L^2(\Omega)^3}. \end{aligned} \tag{33}$$

Hence $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$, which implies the existence of $\mathbf{u} \in H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$ such that $\mathbf{u}_n$ converges to $\mathbf{u}$ strongly in $H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$. Furthermore, $\mathbf{u}$ satisfies $\mathbf{curl}\ \mathbf{u} = \lambda\mathbf{u}$ and $\mathrm{div}\ \mathbf{u} = 0$ in Ω. It follows from Lemma 3 that $\mathbf{u} = 0$ in Ω. However, by (32), we have

$$\|\mathbf{curl}\ \mathbf{u}\|^2_{L^2(\Omega)^3} + \|\lambda\mathbf{u}\|^2_{L^2(\Omega)^3} = 1, \tag{34}$$

which contradicts the fact that $\mathbf{u} = 0$ in Ω. Therefore, the assumption is not true, so the inequality (31) is valid. □

Now we can prove the main results of this subsection.

Proof of Theorem 1. By Hölder inequality and the fact that $\lambda \in L^\infty(\bar{\Omega})$, we know that $a_\lambda(\mathbf{u},\mathbf{v})$ is a symmetric and continuous bilinear form on $H_0(\mathbf{curl};\Omega) \times H_0(\mathbf{curl};\Omega)$ and $b(\mathbf{v},\psi)$ is also a continuous bilinear form on $H_0(\mathbf{curl};\Omega) \times \mathcal{D}$. Define

$$V = \{\mathbf{v} \in H_0(\mathbf{curl};\Omega);\ b(\mathbf{v},\psi) = 0,\ \forall\psi \in \mathcal{D}\}.$$

By Green's formula

$$\int_\Omega \psi\ \mathrm{div}\ \mathbf{v}\ d\mathbf{x} + \int_\Omega \mathbf{v}\cdot\nabla\psi\ d\mathbf{x} = \int_{\partial\Omega} \psi\ \mathbf{v}\cdot\mathbf{n}\ d\mathbf{s},$$

we conclude that

$$V = \{\mathbf{v} \in H_0(\mathbf{curl};\Omega);\ \mathrm{div}\ \mathbf{v} = 0 \text{ in } \Omega\}.$$

On one hand, for any function $\mathbf{v} \in V$, Lemma 4, together with Lemma 3.4 in [11], leads to

$$a_\lambda(\mathbf{v},\mathbf{v}) \geqslant C_1\|\mathbf{curl}\ \mathbf{v}\|^2_{L^2(\Omega)^3} \geqslant C\|\mathbf{v}\|^2_{H(\mathbf{curl};\Omega)}. \tag{35}$$

Hence $a_\lambda(\mathbf{u},\mathbf{v})$ is V-elliptic in the space V. On the other hand, for any function $\psi \in \mathcal{D}$, we have $\nabla\psi \times \mathbf{n} = 0$ on $\partial\Omega$, because $\psi|_{\partial\Omega} = \alpha$ is a constant. Thus, $\nabla\psi \in H_0(\mathbf{curl};\Omega)$. Taking $\mathbf{v} = \nabla\psi$ in the following inequality, we obtain

$$\sup_{\mathbf{v}\in H_0(\mathbf{curl};\Omega)} \frac{b(\mathbf{v},\psi)}{\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}} \geqslant \frac{\int_\Omega \nabla\psi\cdot\nabla\psi\ d\mathbf{x}}{\|\nabla\psi\|_{H(\mathbf{curl};\Omega)}} = \|\nabla\psi\|_{L^2(\Omega)^3}, \tag{36}$$

which implies that $b(\mathbf{v},\psi)$ satisfies the inf-sup condition. According to Babuska-Brezzi Theorem [9], the saddle-point problem (22) admits one and only one solution $(\mathbf{u},\varphi) \in H_0(\mathbf{curl};\Omega) \times \mathcal{D}$. Furthermore, we have

$$\|\mathbf{u}\|_{H(\mathbf{curl};\Omega)} + \|\varphi\|_{H^1(\Omega)} \leqslant C\|\mathbf{f}_\lambda\|_{L^2(\Omega)^3}, \tag{37}$$

where C is a constant independent of $\mathbf{u}$ and φ. □

Proof of Theorem 2. Following the discussion in the beginning of Section 3, we know that the minimization problem (19) is equivalent to the problem (20). Hence, we only need to prove that the problem (20) is equivalent to the saddle-point problem (22).

Assume that $(\mathbf{u}, \varphi) \in H_0(\mathbf{curl}; \Omega) \times \mathcal{D}$ is the solution of the problem (22). Hence $\mathbf{u} \in V$. It follows from Green's formula that

$$\mathbf{u} \in H_0(\mathbf{curl}; \Omega), \quad \text{div } \mathbf{u} = 0 \text{ in } \Omega.$$

For any function $\mathbf{v} \in H_0(\mathbf{curl}; \Omega)$ satisfying div $\mathbf{v} = 0$, we have $\mathbf{v} \in V$. Hence we get

$$a_\lambda(\mathbf{u}, \mathbf{v}) = a_\lambda(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \varphi) = (\mathbf{F}_\lambda, \mathbf{v}),$$

which implies that $\mathbf{u}$ is also the solution of the problem: Find $\mathbf{u} \in V$ such that

$$\int_\Omega (\mathbf{curl}\ \mathbf{u} - \lambda \mathbf{u}) \cdot (\mathbf{curl}\ \mathbf{v} - \lambda \mathbf{v})\ d\mathbf{x} = \int_\Omega \mathbf{f}_\lambda \cdot (\mathbf{curl}\ \mathbf{v} - \lambda \mathbf{v})\ d\mathbf{x},\ \forall \mathbf{v} \in V. \tag{38}$$

Furthermore, by the inequality (35) and Theorem 1.1.2 in [10], we deduce that the function $\mathbf{u}$ is the solution of the problem (20).

Conversely, according to the discussion from p.61 to p.64 of [11], we know that the problem (20) has an unique solution which is just the solution $\mathbf{u}$ of the problem (22) under the condition that $a_\lambda(\mathbf{v}, \mathbf{v}) \geqslant 0$ is valid for every $\mathbf{v} \in H_0(\mathbf{curl}; \Omega)$ and the problem (22) admits one and only one solution. In fact, the condition above is indeed satisfied by the definition of $a_\lambda(\cdot, \cdot)$ and Theorem 1. □

3.2 A numerical method for solving the saddle-point problem (22)

The purpose of this subsection is to introduce a finite element method for solving the saddle-point problem (22). We introduce a standard triangulation $\mathscr{T}_h$ consisting of tetrahedra with the mesh size h. We assume that the triangulation is regular and quasi-uniform. On each element K in $\mathscr{T}_h$, for each positive integer l, we define the space $\mathbb{P}_l(K)$ as the set of all polynomials whose degrees are less than or equal to l. Then the Nédélec edge finite element space [17], of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathscr{T}_h$,

$$X_h = \{\mathbf{v}_h \in H(\mathbf{curl}; \Omega); \mathbf{v}_h|_K \in \mathcal{R}(K), \forall K \in \mathscr{T}_h\},\ X_{h,0} = X_h \cap H_0(\mathbf{curl}; \Omega), \tag{39}$$

where $\mathcal{R}(K)$ stands for the space

$$\mathcal{R}(K) = \{\mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \text{ and } \mathbf{x} \in K\}. \tag{40}$$

Note that the elements of X_h are piecewise polynomial vector fields such that their tangential components are continuous through all faces of each element K in the triangulation $\mathscr{T}_h$.

Moreover, we denote by Y_h the continuous piecewise linear polynomial space associated with the triangulation $\mathscr{T}_h$,

$$Y_h = \{\mu_h \in H^1(\Omega)/\mathbb{R};\ \mu_h|_K \in \mathbb{P}_1(K), \forall K \in \mathscr{T}_h \text{ and } \mu_h|_{\partial\Omega} = \alpha,\ \alpha \in \mathbb{R}\}. \tag{41}$$

In this subsection, we consider the following discrete saddle-point system associated with the saddle-point problem (22): Find $(\mathbf{u}_h, \varphi_h) \in X_{h,0} \times Y_h$ such that

$$\begin{cases} a_\lambda(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \varphi_h) = (\mathbf{F}_\lambda, \mathbf{v}_h), & \forall \mathbf{v}_h \in X_{h,0}, \\ b(\mathbf{u}_h, \psi_h) = 0, & \forall \psi_h \in Y_h. \end{cases} \tag{42}$$

In the following, we prove that the discrete saddle-point problem (42) admits one and only one solution, and the solution of problem (42) approximates to the solution of the saddle-point problem (22).

The following lemma can be proved by a similar manner with the proof of Lemma 4.3 in [3].

Lemma 5. *A function $\mathbf{v}_h \in X_{h,0}$ satisfies $\mathbf{curl}\ \mathbf{v}_h = 0$ if and only if $\mathbf{v}_h$ can be written as the gradient of some function in Y_h.*

□

We fix an element K in $\mathscr{T}_h$ and denote by F_K the affine transformation which maps a reference tetrahedron $\hat{K}$ onto K.

Lemma 6. *Suppose that $\hat{\lambda} \in W^{1,\infty}(\hat{K})$. If a function $\hat{\mathbf{v}}_h \in \mathbb{P}_1(\hat{K})^3$ satisfies*

$$\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = \hat{\lambda}\hat{\mathbf{v}}_h \text{ in } \hat{K}, \tag{43}$$

then $\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = 0$ in $\hat{K}$.

Proof. Let $\hat{\mathbf{v}}_h = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T$, where $\hat{v}_i(\hat{\mathbf{x}}) = \hat{v}_i(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{P}_1(\hat{K})(i = 1, 2, 3)$. The equality (43) leads to

$$\hat{A}\ \hat{\mathbf{v}}_h = 0, \tag{44}$$

where the skew symmetric matrix $\hat{A}$ has the following form

$$\hat{A} = \begin{bmatrix} 0 & \hat{a}_1 & \hat{a}_2 \\ -\hat{a}_1 & 0 & \hat{a}_3 \\ -\hat{a}_2 & -\hat{a}_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \partial_{\hat{x}}\hat{v}_2 - \partial_{\hat{y}}\hat{v}_1 & \partial_{\hat{x}}\hat{v}_3 - \partial_{\hat{z}}\hat{v}_1 \\ \partial_{\hat{y}}\hat{v}_1 - \partial_{\hat{x}}\hat{v}_2 & 0 & \partial_{\hat{y}}\hat{v}_3 - \partial_{\hat{z}}\hat{v}_2 \\ \partial_{\hat{z}}\hat{v}_1 - \partial_{\hat{x}}\hat{v}_3 & \partial_{\hat{z}}\hat{v}_2 - \partial_{\hat{y}}\hat{v}_3 & 0 \end{bmatrix}.$$

Hence, letting $\vartheta(\hat{\mathbf{x}}) = \hat{a}_1\hat{a}_2\hat{a}_3\hat{v}_1\hat{v}_2\hat{v}_3$, we conclude that $\vartheta(\hat{\mathbf{x}}) = 0$ for any point $\hat{\mathbf{x}} \in \hat{K}$. In fact, if it is false, there exists $\hat{\mathbf{x}}_0 \in \hat{K}$ such that $\vartheta(\hat{\mathbf{x}}_0) \neq 0$. Since $\hat{v}_i(\hat{\mathbf{x}}) \in \mathbb{P}_1(\hat{K})$, $\hat{v}_i$

can be written as $\hat{v}_i = a_i\hat{x} + b_i\hat{y} + c_i\hat{z} + d_i$ where $a_i, b_i, c_i, d_i \in \mathbb{R}$ $(i = 1, 2, 3)$. It follows from the equality (43) that $\hat{v}_2 = \alpha\hat{v}_1$ and $\hat{v}_3 = \beta\hat{v}_1$ where α and β are two real numbers. Furthermore, by (43) again, we deduce

$$\frac{1}{\alpha} = \frac{\beta b_1 - \alpha c_1}{c_1 - \beta a_1} \quad \text{and} \quad \frac{1}{\beta} = \frac{\beta b_1 - \alpha c_1}{\alpha a_1 - b_1}, \tag{45}$$

which imply that

$$(1 + \alpha^2 + \beta^2)(\beta b_1 - \alpha c_1) = 0. \tag{46}$$

Thus, we have $\beta b_1 - \alpha c_1 = 0$. This means $\hat{a}_1 = 0$, which contradicts the fact that $\vartheta(\hat{\mathbf{x}}_0) \neq 0$. Therefore, the assumption is not true. Hence $\vartheta(\hat{\mathbf{x}}) = 0$ for any point $\hat{\mathbf{x}} \in \hat{K}$. Moreover, we derive either some $\hat{a}_i = 0$ or some $\hat{v}_i = 0$ because $\vartheta(\hat{\mathbf{x}})$ is a polynomial on $\hat{K}$. In the following, we prove $\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = 0$ for two cases respectively. In one case, there exists some $\hat{a}_i = 0$ and $\hat{v}_1\hat{v}_2\hat{v}_3 \neq 0$; in the other, there exists some $\hat{v}_i = 0$. Firstly, we consider the former. Without loss of generality, we assume $\hat{a}_1 = 0$. Since $\hat{v}_1\hat{v}_2\hat{v}_3 \neq 0$, we have $\hat{v}_3 \neq 0$. In addition, by the equality (44), we obtain

$$\begin{cases} \hat{a}_2\hat{v}_3 = 0, \\ \hat{a}_3\hat{v}_3 = 0, \\ \hat{a}_2\hat{v}_1 + \hat{a}_3\hat{v}_2 = 0. \end{cases} \tag{47}$$

This, together with the fact that $\hat{a}_2$, $\hat{a}_3$ and $\hat{v}_3$ are three polynomials, gives $\hat{a}_2 = 0$ and $\hat{a}_3 = 0$. Therefore, $\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = 0$. Next, we consider the latter. Without loss of generality, we assume $\hat{v}_1 = 0$. By the equality (44) again, we derive

$$\begin{cases} \hat{v}_2\partial_{\hat{x}}\hat{v}_2 + \hat{v}_3\partial_{\hat{x}}\hat{v}_3 = 0, \\ (\partial_{\hat{y}}\hat{v}_3 - \partial_{\hat{z}}\hat{v}_2)\hat{v}_3 = 0, \\ (\partial_{\hat{y}}\hat{v}_3 - \partial_{\hat{z}}\hat{v}_2)\hat{v}_2 = 0, \end{cases} \tag{48}$$

which, together with the fact that $\hat{v}_2$ and $\hat{v}_3$ are two polynomials on $\hat{K}$, yields $\partial_{\hat{x}}\hat{v}_2 = 0$, $\partial_{\hat{x}}\hat{v}_3 = 0$ and $\partial_{\hat{y}}\hat{v}_3 - \partial_{\hat{z}}\hat{v}_2 = 0$. Therefore, $\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = 0$ in $\hat{K}$. □

The following result can be obtained by Lemma 6.

Lemma 7. *Suppose that $\lambda \in W^{1,\infty}(\bar{\Omega})$. There is a constant $C_2 > 0$ independent of h such that for every $\mathbf{v}_h \in X_h$,*

$$\|\mathbf{curl}\ \mathbf{v}_h\|^2_{L^2(\Omega)^3} + \|\lambda\mathbf{v}_h\|^2_{L^2(\Omega)^3} \leqslant C_2\|\mathbf{curl}\ \mathbf{v}_h - \lambda\mathbf{v}_h\|^2_{L^2(\Omega)^3}. \tag{49}$$

Proof. Firstly, we conclude that the following inequality is valid,

$$\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} + \|\hat{\lambda}\hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} \leqslant \hat{C}\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h - \hat{\lambda}\hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3}, \ \forall \hat{\mathbf{v}}_h \in \mathcal{R}(\hat{K}), \tag{50}$$

where the constant $\hat{C}$ is dependent only on $\hat{\lambda}$. We prove the inequality (50) by the reductio ad absurdum. Assume the inequality above is false, we can find a sequence $\{\hat{\mathbf{v}}_h^n\}_{n=1}^{\infty}$ such that, for each positive integer n, $\hat{\mathbf{v}}_h^n \in \mathcal{R}(\hat{K})$ and

$$\left\{ \begin{array}{l} \|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h^n\|^2_{L^2(\hat{K})^3} + \|\hat{\lambda}\hat{\mathbf{v}}_h^n\|^2_{L^2(\hat{K})^3} = 1, \\ \|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h^n - \hat{\lambda}\hat{\mathbf{v}}_h^n\|^2_{L^2(\hat{K})^3} < \frac{1}{n}. \end{array} \right. \tag{51}$$

Therefore, $\|\hat{\lambda}\|_{L^\infty(\hat{K})} \neq 0$. Hence there exist a constant $\sigma > 0$ and a nonempty open set $\tilde{K} \subset \hat{K}$ such that $|\hat{\lambda}| \geqslant \sigma$ on $\tilde{K}$. Furthermore, by (51), we can deduce

$$\sigma^2 \|\hat{\mathbf{v}}_h^n\|^2_{L^2(\tilde{K})^3} \leqslant \|\hat{\lambda}\hat{\mathbf{v}}_h^n\|^2_{L^2(\tilde{K})^3} \leqslant \|\hat{\lambda}\hat{\mathbf{v}}_h^n\|^2_{L^2(\hat{K})^3} < 2(\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h^n\|^2_{L^2(\hat{K})^3} + \frac{1}{n}) \leqslant 2(1 + \frac{1}{n}).$$

Thus, $\|\hat{\mathbf{v}}_h^n\|_{H(\widehat{\mathbf{curl}};\tilde{K})}$ is uniformly bounded. Since $\hat{\mathbf{v}}_h^n$ is a polynomial vector on $\hat{K}$, $\|\hat{\mathbf{v}}_h^n\|_{H(\widehat{\mathbf{curl}};\hat{K})}$ is also uniformly bounded. By the fact that $\hat{\mathbf{v}}_h^n \in \mathcal{R}(\hat{K})$, we can extract a subsequence of $\{\hat{\mathbf{v}}_h^n\}_{n=1}^{\infty}$ which converges strongly in $H(\widehat{\mathbf{curl}}; \hat{K})$. We still denote the subsequence by $\{\hat{\mathbf{v}}_h^n\}_{n=1}^{\infty}$. Hence there exists $\hat{\mathbf{v}}_h \in \mathcal{R}(\hat{K})$ such that $\hat{\mathbf{v}}_h^n$ converges to $\hat{\mathbf{v}}_h$ strongly in $H(\widehat{\mathbf{curl}}; \hat{K})$. In addition, we have $\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} + \|\hat{\lambda}\hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} = 1$, because $\hat{\lambda} \in L^\infty(\hat{K})$. By the fact that $\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h^n - \hat{\lambda}\hat{\mathbf{v}}_h^n\|^2_{L^2(\hat{K})^3} < \frac{1}{n}$, we derive $\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h - \hat{\lambda}\hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} = 0$, which implies that $\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = \hat{\lambda}\hat{\mathbf{v}}_h$ in $\hat{K}$. By Lemma 6, we have $\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h = 0$ and $\hat{\lambda}\hat{\mathbf{v}}_h = 0$ in $\hat{K}$, which contradicts the fact that $\|\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} + \|\hat{\lambda}\hat{\mathbf{v}}_h\|^2_{L^2(\hat{K})^3} = 1$. Hence our assumption is not true.

Next, we prove the inequality (49). By the inequality (50), we obtain

$$\begin{aligned} \|\lambda \mathbf{v}_h\|^2_{L^2(\Omega)^3} &\leqslant Ch \sum_{\hat{K}} \int_{\hat{K}} \left(\hat{\lambda}\hat{\mathbf{v}}_h \cdot \hat{\lambda}\ \hat{\mathbf{v}}_h\right) d\hat{\mathbf{x}} \\ &\leqslant Ch \sum_{\hat{K}} \int_{\hat{K}} (\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h - \hat{\lambda}\hat{\mathbf{v}}_h) \cdot (\widehat{\mathbf{curl}}\ \hat{\mathbf{v}}_h - \hat{\lambda}\hat{\mathbf{v}}_h)\ d\hat{\mathbf{x}} \\ &\leqslant C\|\mathbf{curl}\ \mathbf{v}_h - \lambda \mathbf{v}_h\|^2_{L^2(\Omega)^3}, \end{aligned} \tag{52}$$

where C is a constant independent of h. Furthermore, by (52), we have

$$\|\mathbf{curl}\ \mathbf{v}_h\|^2_{L^2(\Omega)^3} \leqslant 2(\|\mathbf{curl}\ \mathbf{v}_h - \lambda \mathbf{v}_h\|^2_{L^2(\Omega)^3} + \|\lambda \mathbf{v}_h\|^2_{L^2(\Omega)^3}) \leqslant C\|\mathbf{curl}\ \mathbf{v}_h - \lambda \mathbf{v}_h\|^2_{L^2(\Omega)^3},$$

which, together with (52), implies that the inequality (49) is valid. □

Let

$$V_h = \{\mathbf{v}_h \in X_h;\ b(\mathbf{v}_h, \psi_h) = 0,\ \forall \psi_h \in Y_h\} \quad \text{and} \quad V_{h,0} = V_h \cap X_{h,0}.$$

The following result is essentially Proposition 4.12 in [3].

Lemma 8. *There is a constant $C_3 > 0$ independent of h such that for every $\mathbf{v}_h \in V_{h,0}$*

$$\|\mathbf{v}_h\|_{L^2(\Omega)^3} \leqslant C_3 \|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3}. \tag{53}$$

□

Now, we present the existence and uniqueness result.

Theorem 3. *Suppose that $\lambda \in W^{1,\infty}(\bar{\Omega})$ such that either $\lambda = 0$ in $\bar{\Omega}$ or $\lambda \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$. The problem (42) admits one and only one solution $(\mathbf{u}_h, \varphi_h) \in X_{h,0} \times Y_h$. Furthermore,*

$$\|\mathbf{u}_h - \mathbf{u}\|_{H(\mathbf{curl};\Omega)} + \|\varphi_h - \varphi\|_{H^1(\Omega)} \leqslant C\big(\inf_{\mathbf{v}_h \in X_{h,0}} \|\mathbf{v}_h - \mathbf{u}\|_{H(\mathbf{curl};\Omega)} + \inf_{\psi_h \in Y_h} \|\psi_h - \varphi\|_{H^1(\Omega)}\big),$$

where $(\mathbf{u}, \varphi) \in H_0(\mathbf{curl};\Omega) \times \mathcal{D}$ is the solution of the problem (22) and C is a constant independent of h.

Proof. Firstly, it is obvious that $a_\lambda(\mathbf{u}_h, \mathbf{v}_h)$ is a symmetric and continuous bilinear form on $X_{h,0} \times X_{h,0}$ and $b(\mathbf{v}_h, \psi_h)$ is also a continuous bilinear form on $X_{h,0} \times Y_h$. Secondly, for any function $\mathbf{v}_h \in V_{h,0}$, Lemma 7 and Lemma 8 lead to

$$a_\lambda(\mathbf{v}_h, \mathbf{v}_h) = \|\mathbf{curl}\ \mathbf{v}_h - \lambda \mathbf{v}_h\|^2_{L^2(\Omega)^3} \geqslant C\|\mathbf{curl}\ \mathbf{v}_h\|^2_{L^2(\Omega)^3} \geqslant C\|\mathbf{v}_h\|^2_{H(\mathbf{curl};\Omega)}. \tag{54}$$

Hence $a_\lambda(\mathbf{u}_h, \mathbf{v}_h)$ is V-elliptic in the space $V_{h,0}$. Thirdly, for any function $\psi_h \in Y_h$, we obtain $\nabla\psi_h \in X_{h,0}$ due to Lemma 5. Taking $\mathbf{v}_h = \nabla\psi_h$ in the following inequality, we deduce

$$\sup_{\mathbf{v}_h \in X_{h,0}} \frac{b(\mathbf{v}_h, \psi_h)}{\|\mathbf{v}_h\|_{H(\mathbf{curl};\Omega)}} \geqslant \frac{\int_\Omega \nabla\psi_h \cdot \nabla\psi_h d\mathbf{x}}{\|\nabla\psi_h\|_{H(\mathbf{curl};\Omega)}} = \|\nabla\psi_h\|_{L^2(\Omega)^3}, \tag{55}$$

which implies that $b(\mathbf{v}_h, \psi_h)$ satisfies the inf-sup condition. According to Babuska-Brezzi Theorem and Theorem 1, we know that the discrete saddle-point problem (42) admits one and only one solution $(\mathbf{u}_h, \varphi_h) \in X_{h,0} \times Y_h$. Furthermore, we have

$$\|\mathbf{u}_h - \mathbf{u}\|_{H(\mathbf{curl};\Omega)} + \|\varphi_h - \varphi\|_{H^1(\Omega)} \leqslant C\big(\inf_{\mathbf{v}_h \in X_{h,0}} \|\mathbf{v}_h - \mathbf{u}\|_{H(\mathbf{curl};\Omega)} + \inf_{\psi_h \in Y_h} \|\psi_h - \varphi\|_{H^1(\Omega)}\big),$$

where C is a constant independent of h. □

Remark 2. *According to the discussion in this section, the solution of the discrete saddle-point system (42) is an approximate solution of the minimization problem (19).*

4 A solution method of the optimization problem (18)

In this section, we propose a numerical method to solve the minimization problem on λ^* with a known function $\mathbf{u} \in H(\mathbf{curl};\Omega)$,

$$\min_{\forall \lambda^* \in W^{1,\infty}(\bar{\Omega})} E(\mathbf{u}, \lambda^*) = \frac{1}{2}\|\mathbf{curl}\ \mathbf{u} - \lambda^* \mathbf{u}\|^2_{L^2(\Omega)^3}, \tag{56}$$

subject to

$$\lambda^* = \lambda_0 \text{ on } \partial\Omega.$$

Find a function $\tilde{\lambda}_0 \in W^{1,\infty}(\bar{\Omega})$ such that $\tilde{\lambda}_0 = \lambda_0$ on $\partial\Omega$. Set $\lambda = \lambda^* - \tilde{\lambda}_0$ and $\mathbf{g_u} = \tilde{\lambda}_0 \mathbf{u}$. The problem (56) is equivalent to

$$\min_{\forall \lambda \in W^{1,\infty}(\bar{\Omega}) \cap H^1_0(\Omega)} E(\mathbf{u}, \lambda) = \frac{1}{2}\|\mathbf{curl}\ \mathbf{u} - \lambda \mathbf{u} - \mathbf{g_u}\|^2_{L^2(\Omega)^3}. \tag{57}$$

The corresponding discrete form of (57) can be written as

$$\min_{\forall \lambda_h \in Z_{h,0}} E(\mathbf{u}, \lambda_h) = \frac{1}{2}\|\mathbf{curl}\ \mathbf{u} - \lambda_h \mathbf{u} - \mathbf{g_u}\|^2_{L^2(\Omega)^3}, \tag{58}$$

where $Z_{h,0} \subset W^{1,\infty}(\bar{\Omega}) \cap H^1_0(\Omega)$ is a conforming finite element space that consists of continuous piecewise linear polynomials, i.e.,

$$Z_{h,0} = Z_h \cap H^1_0(\Omega), \quad Z_h = \{\phi_h \in W^{1,\infty}(\bar{\Omega}); \phi_h|_K \in \mathbb{P}_1(K), \forall K \in \mathscr{T}_h\}.$$

When $\mathbf{u}$ is known, $E(\mathbf{u}, \lambda_h)$ is a convex quadratic functional on λ_h but not always strictly convex. Let $\langle \cdot, \cdot \rangle$ denote the dual pairing between $Z_{h,0}$ and $Z'_{h,0}$. Define the linear operator $A_h : Z_{h,0} \to Z'_{h,0}$ by

$$\langle A_h \lambda_h, \omega_h \rangle = \int_\Omega |\mathbf{u}|^2 \lambda_h\ \omega_h\ d\mathbf{x}, \quad \forall \lambda_h, \omega_h \in Z_{h,0}.$$

Furthermore, we set $c = \frac{1}{2}\int_\Omega |\mathbf{curl}\ \mathbf{u} - \mathbf{g_u}|^2\ d\mathbf{x}$ and define $g_h \in Z'_{h,0}$ by

$$\langle g_h, \omega_h \rangle = \int_\Omega (\mathbf{curl}\ \mathbf{u} - \mathbf{g_u}) \cdot \mathbf{u}\ \omega_h\ d\mathbf{x}, \quad \forall \omega_h \in Z_{h,0}.$$

With these notations, the minimization problem (58) can be written as

$$\min_{\forall \lambda_h \in Z_{h,0}} E(\mathbf{u}, \lambda_h) = \frac{1}{2}\langle A_h \lambda_h, \lambda_h \rangle - \langle g_h, \lambda_h \rangle + c. \tag{59}$$

4.1 On the minimization problem (59)

In this subsection, we shall prove the problem (59) admits at least one solution. Define

$$\mathcal{K}(A_h) = \{\lambda_h \in Z_{h,0}; \langle A_h\lambda_h, \omega_h\rangle = 0,\ \forall \omega_h \in Z_{h,0}\}.$$

Denote the range of A_h by $\mathcal{R}(A_h)$. Let $P_h : L^2(\Omega) \to Z_{h,0}$ denote the orthogonal projector defined as follows: for every $\xi \in L^2(\Omega)$, $P_h\xi = \xi_h$, where ξ_h belongs to $Z_{h,0}$ and satisfies

$$\int_\Omega \xi_h\ \omega_h\ d\mathbf{x} = \int_\Omega \xi\ \omega_h\ d\mathbf{x},\quad \forall \omega_h \in Z_{h,0}.$$

Lemma 9. *If* $\mathbf{u} \in H(\mathbf{curl};\Omega)$*, then*

$$Z_{h,0} = \mathcal{K}(A_h) \oplus P_h(\mathcal{R}(A_h)), \tag{60}$$

where the symbol $\oplus$ *denotes the direct sum in the sense of the inner product of* $L^2(\Omega)$*.*

Proof. Firstly, we prove

$$\mathcal{K}(A_h) = P_h(\mathcal{R}(A_h))^\perp. \tag{61}$$

For any $\lambda_h \in \mathcal{K}(A_h)$ and $\psi_h \in P_h(\mathcal{R}(A_h))$, there exists a function $\mu_h \in Z_{h,0}$ such that $\psi_h = P_hA_h\mu_h$. Hence we derive

$$\int_\Omega \psi_h\ \lambda_h\ d\mathbf{x} = \langle A_h\mu_h, \lambda_h\rangle = \int_\Omega |\mathbf{u}|^2\lambda_h\ \mu_h\ d\mathbf{x} = \langle A_h\lambda_h, \mu_h\rangle = 0. \tag{62}$$

Thus, $\lambda_h \in P_h(\mathcal{R}(A_h))^\perp$. Conversely, for any $\lambda_h \in P_h(\mathcal{R}(A_h))^\perp$ and $\psi_h \in Z_{h,0}$, we have $P_hA_h\psi_h \in P_h(\mathcal{R}(A_h))$. By the definitions of A_h and P_h, we obtain

$$\langle A_h\lambda_h, \psi_h\rangle = \int_\Omega |\mathbf{u}|^2\lambda_h\ \psi_h\ d\mathbf{x} = \langle A_h\psi_h, \lambda_h\rangle = \int_\Omega P_h(A_h\psi_h)\ \lambda_h\ d\mathbf{x} = 0. \tag{63}$$

As ψ_h is arbitrary, we have $\lambda_h \in \mathcal{K}(A_h)$. Therefore, the equality (61) is valid, which implies that $\mathcal{K}(A_h) \perp P_h(\mathcal{R}(A_h))$.

Next, since $Z_{h,0}$ is a finite dimensional space, $P_h(\mathcal{R}(A_h))$ is a closed subspace of $Z_{h,0}$. Furthermore, we deduce

$$P_h(\mathcal{R}(A_h)) = (P_h(\mathcal{R}(A_h))^\perp)^\perp, \tag{64}$$

which, together with (61), implies

$$Z_{h,0} = \mathcal{K}(A_h) \oplus \mathcal{K}(A_h)^\perp = \mathcal{K}(A_h) \oplus (P_h(\mathcal{R}(A_h))^\perp)^\perp = \mathcal{K}(A_h) \oplus P_h(\mathcal{R}(A_h)).$$

This completes the proof. □

By Lemma 9, we can prove the following result.

Lemma 10. *If* $\mathbf{u} \in H(\mathbf{curl};\Omega)$*, then the problem: Find* $\lambda_h \in Z_{h,0}$ *such that*

$$\langle A_h\lambda_h, \omega_h\rangle = \langle g_h, \omega_h\rangle, \ \forall\omega_h \in Z_{h,0}, \tag{65}$$

admits at least one solution.

Proof. We prove this result by the reductio ad absurdum. Assume the problem (65) is not solvable. Hence we conclude $P_hg_h \neq 0$ and $P_hg_h \in \mathcal{K}(A_h)$. In fact, if $P_hg_h = 0$ or $P_hg_h \notin \mathcal{K}(A_h)$, Lemma 9 implies that $P_hg_h = 0$ or $P_hg_h \in P_h(\mathcal{R}(A_h))$. Therefore, there exists $\psi_h \in Z_{h,0}$ such that $P_hg_h = P_hA_h\psi_h$. This leads to

$$\langle A_h\psi_h, \omega_h\rangle = \int_\Omega P_hg_h \ \omega_h \ d\mathbf{x} = \langle g_h, \omega_h\rangle, \ \forall\omega_h \in Z_{h,0},$$

which contradicts the assumption. Thus, $\langle g_h, P_hg_h\rangle = \|P_hg_h\|^2_{L^2(\Omega)} \neq 0$ and

$$\langle A_hP_hg_h, \omega_h\rangle = 0, \ \forall\omega_h \in Z_{h,0}. \tag{66}$$

Furthermore, letting

$$\mu_h = \frac{c+1}{\langle g_h, P_hg_h\rangle}P_hg_h,$$

we have $\mu_h \in \mathcal{K}(A_h)$. It follows from (66) that

$$E(\mathbf{u}, \mu_h) = \frac{1}{2}\langle A_h\mu_h, \mu_h\rangle - \langle g_h, \mu_h\rangle + c = -(c+1) + c = -1 < 0. \tag{67}$$

This conflicts with the fact that $E(\cdot,\cdot) \geqslant 0$. Therefore, the assumption is not true, so the problem (65) admits at least one solution. □

Lemma 10 implies that there is $\hat\lambda_h \in Z_{h,0}$ such that $P_hA_h\hat\lambda_h = P_hg_h$. Hence,

$$\begin{aligned} E(\mathbf{u}, \lambda_h) &= \frac{1}{2}\langle A_h\lambda_h, \lambda_h\rangle - \langle g_h, \lambda_h\rangle + c \\ &= \frac{1}{2}\langle A_h\lambda_h, \lambda_h\rangle - \langle A_h\hat\lambda_h, \lambda_h\rangle + \frac{1}{2}\langle A_h\hat\lambda_h, \hat\lambda_h\rangle + c - \frac{1}{2}\langle A_h\hat\lambda_h, \hat\lambda_h\rangle \\ &= \frac{1}{2}\langle A_h(\lambda_h - \hat\lambda_h), (\lambda_h - \hat\lambda_h)\rangle + c - \frac{1}{2}\langle A_h\hat\lambda_h, \hat\lambda_h\rangle. \end{aligned} \tag{68}$$

In addition, since $\langle A_h\omega_h, \omega_h\rangle \geqslant 0$ is valid for every $\omega_h \in Z_{h,0}$, we have

$$E(\mathbf{u}, \hat\lambda_h) = \min_{\forall\lambda_h\in Z_{h,0}} E(\mathbf{u}, \lambda_h). \tag{69}$$

So far, we have already proved the following result.

Theorem 4. *The minimization problem (59) admits at least one solution.*

□

4.2 A numerical method for solving the minimization problem (59)

In this subsection, we introduce a conjugate gradient (CG) method to solve the minimization problem (59).

Algorithm 4.1. Give an initial guess λ_h^1, set $g_h^1 := P_h(A_h\lambda_h^1 - g_h)$ and $d_h^1 := -g_h^1$ (without loss of generality, we assume that $d_h^1 \neq 0$). For $k \geqslant 1$, the function λ_h^{k+1} (together with g_h^{k+1} and d_h^{k+1}) is obtained recursively as follows:

Step 1. Compute $\lambda_h^{k+1} := \lambda_h^k + \alpha_k d_h^k$ with

$$\alpha_k := -\frac{\int_\Omega g_h^k \ d_h^k \ d\mathbf{x}}{\langle A_h d_h^k, d_h^k\rangle}.$$

Set $g_h^{k+1} := P_h(A_h\lambda_h^{k+1} - g_h)$. If $\|g_h^{k+1}\|_{L^2(\Omega)} = 0$, then output λ_h^{k+1}; otherwise, goto Step 2.

Step 2. Compute $d_h^{k+1} := -g_h^{k+1} + \beta_k d_h^k$ with

$$\beta_k := \frac{\langle A_h g_h^{k+1}, d_h^k\rangle}{\langle A_h d_h^k, d_h^k\rangle}.$$

In the following, we prove a convergence of the conjugate gradient method.

Theorem 5. *The conjugate gradient method above possesses a finite termination. Namely, there is a positive integer $m \leqslant M^*$ such that*

$$\|g_h^{m+1}\|_{L^2(\Omega)} = 0, \tag{70}$$

where M^* is the dimension of the space $P_h(\mathcal{R}(A_h))$. Moreover, for each positive integer k satisfying $1 \leqslant k \leqslant m$, the following equalities are valid,

$$\int_\Omega g_h^k \ d_h^k \ d\mathbf{x} = -\|g_h^k\|^2_{L^2(\Omega)}, \tag{71}$$

$$\int_\Omega g_h^k \ d_h^j \ d\mathbf{x} = 0, \quad \int_\Omega g_h^k \ g_h^j \ d\mathbf{x} = 0, \quad j = 1, 2, \cdots, k-1, \tag{72}$$

$$\langle A_h d_h^k, d_h^j\rangle = 0, \quad \langle A_h g_h^{k+1}, d_h^j\rangle = 0, \quad j = 1, 2, \cdots, k-1. \tag{73}$$

Proof. We can prove the equalities (71)-(73) by the induction principle. By Lemma 10, we know that there is $\hat{\lambda}_h \in Z_{h,0}$ such that $P_h A_h \hat{\lambda}_h = P_h g_h$. Furthermore, we have

$$g_h^{k+1} = P_h A_h \lambda_h^{k+1} - P_h g_h = P_h A_h(\lambda_h^{k+1} - \hat{\lambda}_h) \in P_h(\mathcal{R}(A_h)), \tag{74}$$

which, together with (72), implies that there exists a positive integer $m \leqslant M^*$ such that

$$\|g_h^{m+1}\|_{L^2(\Omega)} = 0.$$

Hence, the conjugate gradient method above possesses a finite termination. □

Theorem 6. *If* $\|g_h^{m+1}\|_{L^2(\Omega)} = 0$*, then* λ_h^{m+1} *is the solution of the minimization problem (59).*

Proof. If $\|g_h^{m+1}\|_{L^2(\Omega)} = 0$, we have $P_h A_h \lambda_h^{m+1} = P_h g_h$. This implies

$$\langle A_h \lambda_h^{m+1}, \omega_h \rangle = \int_\Omega P_h g_h \; \omega_h \; d\mathbf{x} = \langle g_h, \omega_h \rangle, \; \forall \omega_h \in Z_{h,0}.$$

Hence λ_h^{m+1} is the solution of the problem (65). The argument used in proving (69) shows that

$$E(\mathbf{u}, \lambda_h^{m+1}) = \min_{\forall \lambda_h \in Z_{h,0}} E(\mathbf{u}, \lambda_h). \tag{75}$$

This completes the proof. □

Remark 3. *Theorem 5 and Theorem 6 imply that when the conjugate gradient method terminates, the resulting* λ_h^{m+1} *is a solution of (59).*

5 A new iterative method for solving the Beltrami field equations

As pointed out in Section 2, in order to solve the Beltrami field equations (11), we would like to solve the problems (17) and (18) alternatively. Furthermore, the approximate solutions of the problems (17) and (18) can be obtained by solving the discrete saddle-point problem (42) and the minimization problem (58) respectively (refer to Section 3 and Section 4 for the details). Naturally, an iterative algorithm for solving the Beltrami field equations can be described as follows:

Algorithm 5.1. Give the initial guess $\mathbf{u}_h^0 := 0$ and $\lambda_h^0 := \bar{\lambda}_h^0 := 0$. Find a function $\lambda_{h,0} \in Z_h$ such that $\lambda_{h,0} = \lambda_0$ on $\partial\Omega$. For $n \geqslant 1$, $\mathbf{u}_h^n$ and λ_h^n are obtained by the following two steps.

Step 1. Compute $(\mathbf{u}_h^n, \varphi_h^n) \in X_h \times Y_h$ such that, for $\forall \mathbf{v}_h \in X_{h,0}$ and $\forall \psi_h \in Y_h$,

$$\begin{cases} \int_\Omega (\mathbf{curl}\; \mathbf{u}_h^n - \lambda_h^{n-1} \mathbf{u}_h^n) \cdot (\mathbf{curl}\; \mathbf{v}_h - \lambda_h^{n-1} \mathbf{v}_h) \; d\mathbf{x} + \int_\Omega \nabla \varphi_h^n \cdot \mathbf{v}_h \; d\mathbf{x} = 0, \\ \int_\Omega \mathbf{u}_h^n \cdot \nabla \psi_h \; d\mathbf{x} = 0, \\ \mathbf{u}_h^n \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} \text{ on } \partial\Omega. \end{cases}$$

Step 2. Compute $\lambda_h^n = \bar{\lambda}_h^n + \lambda_{h,0} \in Z_h$, where $\bar{\lambda}_h^n \in Z_{h,0}$ is obtained by solving the following minimization problem with **Algorithm** 4.1, in which the initial guess is chosen as $\bar{\lambda}_h^{n-1}$,

$$\bar{\lambda}_h^n := \arg \min_{\forall \bar{\lambda}_h \in Z_{h,0}} \frac{1}{2} \|\mathbf{curl}\; \mathbf{u}_h^n - (\bar{\lambda}_h + \lambda_{h,0}) \mathbf{u}_h^n\|^2_{L^2(\Omega)^3}.$$

For a given small number $\varepsilon > 0$, when $\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{L^2(\Omega)^3} + \|\lambda_h^n - \lambda_h^{n-1}\|_{L^2(\Omega)} < \varepsilon$, the iteration terminates.

Remark 4. *From the iterative method above, we know that either* $\lambda_h^{n-1} = 0$ *in* $\bar{\Omega}$ *or* $\lambda_h^{n-1}|_\Sigma = \lambda_0|_\Sigma \neq 0$ *where* Σ *is a nonempty open subset of* $\partial\Omega$*. This property of* λ_h^{n-1} *was used in the discussion of Section 3.*

Remark 5. *The saddle-point system in Step 1 can be solved by existing methods such as Uzawa-type algorithms (refer to [14], [15] and [20]). In Step 2, CG method is applied to obtain* $\bar{\lambda}_h^n$ *so that one can avoid to considering a troublesome hyperbolic system which has to be solved in the Grad-Rubin scheme (see [21]).*

In the following, we verify the convergence of the iterative method above. Choosing $\lambda = 0$ in the definition of $a_\lambda(\mathbf{u}, \mathbf{v})$, we get

$$a_0(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \mathbf{v}\ d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in H(\mathbf{curl}; \Omega).$$

By a similar manner in the proofs of Theorem 1 and Theorem 3, we can verify the following two results directly.

Lemma 11. *The problem: Find* $\mathbf{u} \in H(\mathbf{curl}; \Omega)$ *and* $\varphi \in \mathcal{D}$ *such that*

$$\begin{cases} a_0(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \varphi) = 0, & \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega), \\ b(\mathbf{u}, \psi) = 0, & \forall \psi \in \mathcal{D}, \\ \mathbf{u} \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} & on\ \partial\Omega, \end{cases} \tag{76}$$

admits one and only one solution $(\tilde{\mathbf{u}}, \tilde{\varphi}) \in H(\mathbf{curl}; \Omega) \times \mathcal{D}$*. Moreover,*

$$\|\tilde{\mathbf{u}}\|_{H(\mathbf{curl};\Omega)} + \|\tilde{\varphi}\|_{H^1(\Omega)} \leqslant C\|\mathbf{u}_0 \times \mathbf{n}\|_{\mathcal{X}_{\partial\Omega}}. \tag{77}$$

□

Lemma 12. $(\mathbf{u}_h^1, \varphi_h^1) \in X_h \times Y_h$ *obtained by* **Algorithm** *5.1, is the solution of the problem: Find* $\mathbf{u}_h \in X_h$ *and* $\varphi_h \in Y_h$ *such that*

$$\begin{cases} a_0(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \varphi_h) = 0, & \forall \mathbf{v}_h \in X_{h,0}, \\ b(\mathbf{u}_h, \psi_h) = 0, & \forall \psi_h \in Y_h, \\ \mathbf{u}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} & on\ \partial\Omega. \end{cases} \tag{78}$$

Moreover, the following inequality is valid for $\forall \psi_h \in Y_h$ *and* $\forall \mathbf{w}_h \in X_h$ *such that* $\mathbf{w}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ *on* $\partial\Omega$*,*

$$\|\mathbf{u}_h^1 - \tilde{\mathbf{u}}\|_{H(\mathbf{curl};\Omega)} + \|\varphi_h^1 - \tilde{\varphi}\|_{H^1(\Omega)} \leqslant C\big(\|\mathbf{w}_h - \tilde{\mathbf{u}}\|_{H(\mathbf{curl};\Omega)} + \|\psi_h - \tilde{\varphi}\|_{H^1(\Omega)}\big), \tag{79}$$

where C *is a constant independent of* h*.*

□

By Lemma 11 and Lemma 12, we can prove the following result.

Lemma 13. *There is a constant C independent of h such that*

$$\|\mathbf{v}_h\|_{L^2(\Omega)^3} \leqslant C\big(\|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\mathbf{u}_0 \times \mathbf{n}\|_{\mathcal{X}_{\partial\Omega}}\big), \tag{80}$$

is valid for every $\mathbf{v}_h \in V_h$ satisfying $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$.

Proof. When $\mathbf{u}_0 \times \mathbf{n} = 0$ on $\partial\Omega$, Lemma 8 yields the inequality (80). When $\mathbf{u}_0 \times \mathbf{n} \neq 0$ on $\partial\Omega$, it follows from the inequalities (77) and (79) that

$$\|\mathbf{u}_h^1\|_{H(\mathbf{curl};\Omega)} + \|\varphi_h^1\|_{H^1(\Omega)} \leqslant C\|\mathbf{u}_0 \times \mathbf{n}\|_{\mathcal{X}_{\partial\Omega}}, \tag{81}$$

where C is a constant independent of h. For every $\mathbf{v}_h \in V_h$ satisfying $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$, letting $\mathbf{w}_h = \mathbf{v}_h - \mathbf{u}_h^1$, we have $\mathbf{w}_h \in V_{h,0}$. By the triangle inequality and (53), we derive

$$\|\mathbf{v}_h\|_{L^2(\Omega)^3} \leqslant \|\mathbf{w}_h\|_{L^2(\Omega)^3} + \|\mathbf{u}_h^1\|_{L^2(\Omega)^3} \leqslant C\|\mathbf{curl}\ \mathbf{w}_h\|_{L^2(\Omega)^3} + \|\mathbf{u}_h^1\|_{L^2(\Omega)^3}. \tag{82}$$

In addition, noting that $\mathbf{w}_h = \mathbf{v}_h - \mathbf{u}_h^1$, we obtain

$$\|\mathbf{curl}\ \mathbf{w}_h\|_{L^2(\Omega)^3} \leqslant \|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{u}_h^1\|_{L^2(\Omega)^3}, \tag{83}$$

which, together with (81) and (82), leads to

$$\begin{aligned} \|\mathbf{v}_h\|_{L^2(\Omega)^3} &\leqslant C(\|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\mathbf{u}_h^1\|_{H(\mathbf{curl};\Omega)}) \\ &\leqslant C(\|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\mathbf{u}_0 \times \mathbf{n}\|_{\mathcal{X}_{\partial\Omega}}), \end{aligned} \tag{84}$$

where C is a constant independent of h. □

Lemma 13 implies the following lemma which provides us an useful inequality.

Lemma 14. *There is a constant C such that*

$$\|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\lambda_h \mathbf{v}_h\|_{L^2(\Omega)^3} \leqslant C\|\mathbf{curl}\ \mathbf{v}_h - \lambda_h \mathbf{v}_h\|_{L^2(\Omega)^3}, \tag{85}$$

is valid for every $\lambda_h \in Z_h$ and $\mathbf{v}_h \in V_h$ which satisfies that $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$.

Proof. We prove the inequality (85) by the reductio ad absurdum. Assume the inequality above is false. Hence we can find two sequences $\{\mathbf{v}_h^n\}_{n=1}^{\infty}$ and $\{\lambda_h^n\}_{n=1}^{\infty}$ such that, for each positive integer n, $\mathbf{v}_h^n \in V_h$, $\lambda_h^n \in Z_h$ and

$$\begin{cases} \|\mathbf{curl}\ \mathbf{v}_h^n\|_{L^2(\Omega)^3} + \|\lambda_h^n \mathbf{v}_h^n\|_{L^2(\Omega)^3} = 1, \\ \|\mathbf{curl}\ \mathbf{v}_h^n - \lambda_h^n \mathbf{v}_h^n\|_{L^2(\Omega)^3} < \frac{1}{n}, \\ \mathbf{v}_h^n \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} \text{ on } \partial\Omega. \end{cases} \tag{86}$$

Note that $\mathbf{v}_h^n \in V_h$ and $\mathbf{v}_h^n \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$. Lemma 13 leads to

$$\|\mathbf{v}_h^n\|_{L^2(\Omega)^3} \leqslant C(\|\mathbf{curl}\ \mathbf{v}_h^n\|_{L^2(\Omega)^3} + \|\mathbf{u}_0 \times \mathbf{n}\|_{\mathcal{X}_{\partial\Omega}}), \tag{87}$$

where C is a constant independent of $\mathbf{v}_h^n$ and h. This, together with (86), implies that the sequence $\{\mathbf{v}_h^n\}_{n=1}^{\infty}$ is uniformly bounded in $H(\mathbf{curl};\Omega)$. Furthermore, by the fact that $\mathbf{v}_h^n \in V_h$, we can extract a subsequence of $\{\mathbf{v}_h^n\}_{n=1}^{\infty}$ which converges strongly in $L^2(\Omega)^3$. We still denote the subsequence by $\{\mathbf{v}_h^n\}_{n=1}^{\infty}$. Thus,

$$\|\mathbf{curl}\ \mathbf{v}_h^n - \mathbf{curl}\ \mathbf{v}_h^m\|_{L^2(\Omega)^3} \leqslant C\frac{1}{h}\|\mathbf{v}_h^n - \mathbf{v}_h^m\|_{L^2(\Omega)^3}, \tag{88}$$

which means that the sequence $\{\mathbf{v}_h^n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $H(\mathbf{curl};\Omega)$. Hence there exists a function $\mathbf{v}_h \in H(\mathbf{curl};\Omega)$ such that $\mathbf{v}_h^n$ converges to $\mathbf{v}_h$ strongly in $H(\mathbf{curl};\Omega)$. In addition, it follows from (86) that

$$\begin{aligned}\|\lambda_h^n\mathbf{u}_h^n - \lambda_h^m\mathbf{u}_h^m\|_{L^2(\Omega)^3} &\leqslant \|\mathbf{curl}\ \mathbf{v}_h^n - \lambda_h^n\mathbf{u}_h^n\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{v}_h^m - \lambda_h^m\mathbf{u}_h^m\|_{L^2(\Omega)^3}\\ &\quad + \|\mathbf{curl}\ \mathbf{v}_h^n - \mathbf{curl}\ \mathbf{v}_h^m\|_{L^2(\Omega)^3}\\ &\leqslant \frac{1}{n} + \frac{1}{m} + \|\mathbf{curl}\ \mathbf{v}_h^n - \mathbf{curl}\ \mathbf{v}_h^m\|_{L^2(\Omega)^3}.\end{aligned} \tag{89}$$

The inequality (89) implies that the sequence $\{\lambda_h^n\mathbf{v}_h^n\}_{n=1}^{\infty}$ converges strongly in $L^2(\Omega)^3$. In the following, we prove that there exists a function $\lambda_h \in Z_h$ such that $\lambda_h^n\mathbf{v}_h^n$ converges to $\lambda_h\mathbf{v}_h$ strongly in $L^2(\Omega)^3$. Firstly, we suppose that $\{\lambda_h^n\}_{n=1}^{\infty}$ is uniformly bounded in $L^{\infty}(\Omega)$. This yields that there is a constant $M > 0$ such that $\|\lambda_h^n\|_{L^{\infty}(\Omega)} \leqslant M$. Furthermore, since λ_h^n is a piecewise linear polynomial, there exists a subsequence of $\{\lambda_h^n\}_{n=1}^{\infty}$ which is a Cauchy sequence in $L^{\infty}(\Omega)$. We still denote the subsequence by $\{\lambda_h^n\}_{n=1}^{\infty}$. Hence there is a piecewise linear polynomial $\lambda_h \in Z_h$ such that λ_h^n converges to λ_h strongly in $L^{\infty}(\Omega)$. Therefore,

$$\|\lambda_h^n\mathbf{v}_h^n - \lambda_h\mathbf{v}_h\|_{L^2(\Omega)^3} \leqslant M\|\mathbf{v}_h^n - \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\mathbf{v}_h\|_{L^2(\Omega)^3}\|\lambda_h^n - \lambda_h\|_{L^{\infty}(\Omega)}. \tag{90}$$

This gives that $\lambda_h^n\mathbf{v}_h^n$ converges to $\lambda_h\mathbf{v}_h$ strongly in $L^2(\Omega)^3$. Next, we suppose that λ_h^n is not uniformly bounded in $L^{\infty}(\Omega)$. It means that there are a few elements $\{K_i\}_{i=1}^{j}$ in $\mathscr{T}_h$ such that, for each i $(1 \leqslant i \leqslant j)$, $\{\lambda_h^n\}_{n=1}^{\infty}$ is not uniformly bounded in $L^{\infty}(K_i)$, while $\{\lambda_h^n\}_{n=1}^{\infty}$ is uniformly bounded in $L^{\infty}(U)$ where $U = \mathscr{T}_h \setminus \bigcup_{i=1}^{j} K_i$. On one hand, by the proof above, we can deduce that there is a continuous piecewise linear polynomial $\lambda_h \in L^{\infty}(U)$ such that $\lambda_h^n\mathbf{v}_h^n \to \lambda_h\mathbf{v}_h$ strongly in $L^2(U)^3$, because $\{\lambda_h^n\}_{n=1}^{\infty}$ is uniformly bounded in $L^{\infty}(U)$. On the other hand, since $\{\lambda_h^n\}_{n=1}^{\infty}$ is not uniformly bounded in $L^{\infty}(K_i)$ $(1 \leqslant i \leqslant j)$, there exists a subsequence $\{\lambda_h^{n_k}\}_{k=1}^{\infty}$ of $\{\lambda_h^n\}_{n=1}^{\infty}$ such that $\|\lambda_h^{n_k}\|_{L^{\infty}(K_i)} \to \infty$ as $k \to \infty$. By the fact that $\|\lambda_h^n\mathbf{v}_h^n\|_{L^2(\Omega)^3} \leqslant 1$, we see $\|\mathbf{v}_h^{n_k}\|_{L^2(K_i)^3} \to 0$ as $k \to \infty$. Since

$\{\mathbf{v}_h^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $H(\mathbf{curl};\Omega)$, we have $\mathbf{v}_h = 0$ in K_i and $\|\mathbf{v}_h^n\|_{L^2(K_i)^3} \to 0$ as $n \to \infty$. By (86), we obtain

$$\|\lambda_h^n \mathbf{v}_h^n\|_{L^2(K_i)^3} \leqslant \|\mathbf{curl}\ \mathbf{v}_h^n\|_{L^2(K_i)^3} + \frac{1}{n} \leqslant C\frac{1}{h}\|\mathbf{v}_h^n\|_{L^2(K_i)^3} + \frac{1}{n} \to 0, \text{ as } n \to \infty. \quad (91)$$

Extending λ_h from U to $\bigcup_{i=1}^{j} K_i$ by any continuous piecewise linear polynomial, we still denote the extension by λ_h. It follows from (89) that $\lambda_h^n \mathbf{v}_h^n$ converges to $\lambda_h \mathbf{v}_h$ strongly in $L^2(\Omega)^3$. According to the discussion above, we conclude that there exists a function $\lambda_h \in Z_h$ such that $\lambda_h^n \mathbf{v}_h^n$ converges to $\lambda_h \mathbf{v}_h$ strongly in $L^2(\Omega)^3$. Therefore, by (86) again, we derive

$$\begin{cases} \|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3} + \|\lambda_h \mathbf{v}_h\|_{L^2(\Omega)^3} = 1, \\ \mathbf{curl}\ \mathbf{v}_h - \lambda_h \mathbf{v}_h = 0. \end{cases} \quad (92)$$

Lemma 7 and the second equality of (92) imply that $\mathbf{curl}\ \mathbf{v}_h = 0$ and $\lambda_h \mathbf{v}_h = 0$, which contradicts the first equality of (92). Hence our assumption is not true, so the inequality (85) is valid. □

By Lemma 14, we can obtain an important property of the sequence $\{\mathbf{u}_h^n\}_{n=0}^{\infty}$, which plays a key role in the proof of the convergence of **Algorithm** 5.1.

Lemma 15. *The sequence $\{\mathbf{u}_h^n\}_{n=0}^{\infty}$ is uniformly bounded in $H(\mathbf{curl};\Omega)$.*

Proof. For each $n \geqslant 1$, by Theorem 2, we know that $\mathbf{u}_h^n$ satisfies

$$\mathbf{u}_h^n = \arg \min_{\forall \mathbf{u}_h \in X_h} E(\mathbf{u}_h, \lambda_h^{n-1}) = \frac{1}{2}\|\mathbf{curl}\ \mathbf{u}_h - \lambda_h^{n-1}\mathbf{u}_h\|_{L^2(\Omega)^3}^2, \quad (93)$$

subject to

$$b(\mathbf{u}_h, \psi_h) = 0, \quad \forall \psi_h \in Y_h, \quad \text{and} \quad \mathbf{u}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} \text{ on } \partial\Omega.$$

Furthermore, since $\lambda_h^n = \bar{\lambda}_h^n + \lambda_{h,0}$ where $\bar{\lambda}_h^n$ is the solution of the minimization problem in Step 2 of **Algorithm** 5.1, we have

$$E(\mathbf{u}_h^n, \lambda_h^{n-1}) \leqslant E(\mathbf{u}_h^{n-1}, \lambda_h^{n-1}) \leqslant E(\mathbf{u}_h^{n-1}, \lambda_h^{n-2}) \leqslant \cdots \leqslant E(\mathbf{u}_h^1, \lambda_h^1). \quad (94)$$

Lemma 14, together with (94), implies

$$\|\mathbf{curl}\ \mathbf{u}_h^n\|_{L^2(\Omega)^3}^2 \leqslant CE(\mathbf{u}_h^n, \lambda_h^{n-1}) \leqslant CE(\mathbf{u}_h^1, \lambda_h^1). \quad (95)$$

It follows from the inequalities (80) and (95) that

$$\|\mathbf{u}_h^n\|_{H(\mathbf{curl};\Omega)}^2 \leqslant C(E(u_h^1, \lambda_h^1) + \|\mathbf{u}_0 \times \mathbf{n}\|_{\mathcal{X}_{\partial\Omega}}^2), \quad (96)$$

where C is a constant independent of λ_h^k and $\mathbf{u}_h^k$ $(1 \leqslant k \leqslant n)$. Therefore, the sequence $\{\mathbf{u}_h^n\}_{n=0}^{\infty}$ is uniformly bounded in $H(\mathbf{curl};\Omega)$. □

Now we can prove the convergence of **Algorithm** 5.1.

Theorem 7. *The sequence* $\{(\mathbf{u}_h^n, \lambda_h^n)\}_{n=0}^{\infty} \subset X_h \times Z_h$ *obtained by* **Algorithm** *5.1 is convergent. This means that there is* $(\mathbf{u}_h^*, \lambda_h^*) \in X_h \times Z_h$ *such that* $\mathbf{u}_h^n$ *converges to* $\mathbf{u}_h^*$ *strongly in* $H(\mathbf{curl};\Omega)$ *and* λ_h^n *converges to* λ_h^* *strongly in* $W^{1,\infty}(\bar{\Omega})$.

Proof. According to **Algorithm** 5.1, for each $n \geqslant 1$, we have $\lambda_h^n = \bar{\lambda}_h^n + \lambda_{h,0}$, where $\bar{\lambda}_h^n$ is the solution of the minimization problem in Step 2 of **Algorithm** 5.1, and $\lambda_{h,0} \in Z_h$ satisfies $\lambda_{h,0} = \lambda_0$ on $\partial\Omega$. Finding a function $\mathbf{u}_{h,0} \in V_h$ such that $\mathbf{u}_{h,0} \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$, and letting $\bar{\mathbf{u}}_h^n = \mathbf{u}_h^n - \mathbf{u}_{h,0}$, we have $\bar{\mathbf{u}}_h^n \in V_{h,0}$. Hence, the objective functional $E(\mathbf{u}_h, \lambda_h)$ can be regarded as the functional $E(\bar{\mathbf{u}}_h, \bar{\lambda}_h)$ with the variable $(\bar{\mathbf{u}}_h, \bar{\lambda}_h) \in V_{h,0} \times Z_{h,0}$. By Theorem 2, we deduce that $\mathbf{u}_h^n$ satisfies the equation (93). Thus, we get

$$\bar{\mathbf{u}}_h^n = \arg\min_{\forall \bar{\mathbf{u}}_h \in V_{h,0}} E(\bar{\mathbf{u}}_h, \bar{\lambda}_h^{n-1}) = \frac{1}{2}\|\mathbf{curl}\ (\bar{\mathbf{u}}_h + \mathbf{u}_{h,0}) - (\bar{\lambda}_h^{n-1} + \lambda_{h,0})(\bar{\mathbf{u}}_h + \mathbf{u}_{h,0})\|_{L^2(\Omega)^3}^2.$$

Hence $\partial_{\bar{\mathbf{u}}_h} E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1}) \cdot \bar{\mathbf{v}}_h = 0$ is valid for any $\bar{\mathbf{v}}_h \in V_{h,0}$. Furthermore, for any $(\bar{\mathbf{v}}_h, \bar{\omega}_h) \in V_{h,0} \times Z_{h,0}$, we derive

$$\nabla E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1}) \begin{pmatrix} \bar{\mathbf{v}}_h \\ \bar{\omega}_h \end{pmatrix} = \begin{pmatrix} \partial_{\bar{\mathbf{u}}_h} E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1}) \\ \partial_{\bar{\lambda}_h} E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1}) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{v}}_h \\ \bar{\omega}_h \end{pmatrix} = \partial_{\bar{\lambda}_h} E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1}) \cdot \bar{\omega}_h. \quad (97)$$

According to the discussion in Section 4, we know that $\partial_{\bar{\lambda}_h} E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h) \cdot \bar{\omega}_h$ can be written as

$$\partial_{\bar{\lambda}_h} E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h) \cdot \bar{\omega}_h = \langle A_{h,n} \bar{\lambda}_h, \bar{\omega}_h \rangle - \langle g_{h,n}, \bar{\omega}_h \rangle, \quad (98)$$

where the linear operator $A_{h,n} : Z_{h,0} \to Z'_{h,0}$ is defined by

$$\langle A_{h,n} \bar{\lambda}_h, \bar{\omega}_h \rangle = \int_\Omega |\mathbf{u}_h^n|^2 \bar{\lambda}_h\ \bar{\omega}_h\ d\mathbf{x}, \quad \forall \bar{\lambda}_h, \bar{\omega}_h \in Z_{h,0},$$

and $g_{h,n} \in Z'_{h,0}$ is defined as follows

$$\langle g_{h,n}, \bar{\omega}_h \rangle = \int_\Omega (\mathbf{curl}\ \mathbf{u}_h^n - \lambda_{h,0} \mathbf{u}_h^n) \cdot \mathbf{u}_h^n\ \bar{\omega}_h\ d\mathbf{x}, \quad \forall \bar{\omega}_h \in Z_{h,0}.$$

Lemma 15 and the fact that $\mathbf{u}_h^n \in V_h$ imply that the sequence $\{\mathbf{u}_h^n\}_{n=0}^{\infty}$ is also uniformly bounded in $L^\infty(\Omega)^3$. Hence there exists a constant $M > 0$ such that

$$\langle A_{h,n} \bar{\lambda}_h, \bar{\omega}_h \rangle \leqslant M \|\bar{\lambda}_h\|_{L^2(\Omega)} \|\bar{\omega}_h\|_{L^2(\Omega)}, \quad \forall \bar{\lambda}_h, \bar{\omega}_h \in Z_{h,0}. \quad (99)$$

Set $F_n(\bar{\lambda}_h) = E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h)$. By Taylor's formula, together with (99), we deduce, for any real number $\alpha \in (0, \alpha^*]$ with $\alpha^* = \|\bar{d}_h\|_{L^2(\Omega)}^2 / \langle A_{h,2} \bar{d}_h, \bar{d}_h \rangle$, that

$$F_2(\bar{\lambda}_h^1 + \alpha \bar{d}_h) \leqslant F_2(\bar{\lambda}_h^1) + \alpha \langle A_{h,2} \bar{\lambda}_h^1 - g_{h,2}, \bar{d}_h \rangle + \frac{\alpha^2}{2} M \|\bar{d}_h\|_{L^2(\Omega)}^2, \quad (100)$$

where $\bar{d}_h = -P_h(A_{h,2}\bar{\lambda}_h^1 - g_{h,2})$. Since $\bar{\lambda}_h^2$ is the solution of the minimization problem in Step 2 of **Algorithm** 5.1, we have

$$F_2(\bar{\lambda}_h^2) \leqslant F_2(\bar{\lambda}_h^1 + \alpha\bar{d}_h), \tag{101}$$

which, together with the inequality (100), implies

$$F_2(\bar{\lambda}_h^2) \leqslant F_2(\bar{\lambda}_h^1 + \alpha\bar{d}_h) \leqslant F_2(\bar{\lambda}_h^1) + \alpha\langle A_{h,2}\bar{\lambda}_h^1 - g_{h,2}, \bar{d}_h\rangle + \frac{\alpha^2}{2}M\|\bar{d}_h\|^2_{L^2(\Omega)}. \tag{102}$$

Taking $\alpha = -\langle A_{h,2}\bar{\lambda}_h^1 - g_{h,2}, \bar{d}_h\rangle/(M\|\bar{d}_h\|^2_{L^2(\Omega)}) \in (0, \alpha^*]$ in (102), we obtain

$$F_2(\bar{\lambda}_h^1) - F_2(\bar{\lambda}_h^2) \geqslant \frac{1}{2M}\|A_{h,2}\bar{\lambda}_h^1 - g_{h,2}\|^2_{L^2(\Omega)}. \tag{103}$$

Similarly, we have

$$F_n(\bar{\lambda}_h^{n-1}) - F_n(\bar{\lambda}_h^n) \geqslant \frac{1}{2M}\|A_{h,n}\bar{\lambda}_h^{n-1} - g_{h,n}\|^2_{L^2(\Omega)}. \tag{104}$$

By the inequality (94), we obtain $F_n(\bar{\lambda}_h^{n-1}) \leqslant F_{n-1}(\bar{\lambda}_h^{n-1})$, which leads to

$$F_2(\bar{\lambda}_h^1) - F_n(\bar{\lambda}_h^n) \geqslant \frac{1}{2M}\sum_{k=2}^{n}\|A_{h,k}\bar{\lambda}_h^{k-1} - g_{h,k}\|^2_{L^2(\Omega)}. \tag{105}$$

Therefore, we get

$$\|A_{h,n}\bar{\lambda}_h^{n-1} - g_{h,n}\|^2_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

which, together with Hölder inequality and (98), implies

$$\sup_{\substack{\forall\bar{\omega}_h\in Z_{h,0}\\ \|\bar{\omega}_h\|_{H^1(\Omega)}=1}} |\partial_{\bar{\lambda}_h}E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1})\cdot\bar{\omega}_h| \leqslant \|A_{h,n}\bar{\lambda}_h^{n-1} - g_{h,n}\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty. \tag{106}$$

When $n \to \infty$, it follows from (97) and (106) that, for $\forall\bar{\mathbf{v}}_h \in V_{h,0}$ and $\forall\bar{\omega}_h \in Z_{h,0}$,

$$\sup_{\substack{\|\bar{\mathbf{v}}_h\|_{H(\mathbf{curl};\Omega)}=1\\ \|\bar{\omega}_h\|_{H^1(\Omega)}=1}} \left|\nabla E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1})\begin{pmatrix}\bar{\mathbf{v}}_h\\ \bar{\omega}_h\end{pmatrix}\right| = \sup_{\|\bar{\omega}_h\|_{H^1(\Omega)}=1} |\partial_{\bar{\lambda}_h}E(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1})\cdot\bar{\omega}_h| \to 0.$$

Hence $\{(\bar{\mathbf{u}}_h^n, \bar{\lambda}_h^{n-1})\}_{n=0}^\infty$ is a convergent sequence, which implies that $\{(\mathbf{u}_h^n, \lambda_h^n)\}_{n=0}^\infty$ is also convergent. □

6 The decay of the objective functional with the mesh size decreasing

In this section, we derive an estimate of $E(\mathbf{u}_h^*, \lambda_h^*)$ where $(\mathbf{u}_h^*, \lambda_h^*) \in V_h \times Z_h$ is the limit generated by **Algorithm** 5.1 (see Theorem 7 for the details). As we will see, this estimate implies that $(\mathbf{u}_h^*, \lambda_h^*)$ is a good approximation of $(\mathbf{u}, \lambda)$ which is some solution of the Beltrami field problem (11).

First of all, we give several auxiliary results.

Lemma 16. *For every function $\mathbf{v}_h \in V_h$ such that $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$, the following inequality is valid,*

$$\|\mathbf{curl}\ \mathbf{u}_h^1\|_{L^2(\Omega)^3} \leqslant \|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3}. \tag{107}$$

Proof. Note that $\lambda_h^0 = 0$. By Theorem 2, we know that $\mathbf{u}_h^1$ satisfies

$$\mathbf{u}_h^1 = \arg \min_{\forall \mathbf{u}_h \in V_h} E(\mathbf{u}_h, \lambda_h^0) = E(\mathbf{u}_h, 0) \text{ subject to } \mathbf{u}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} \text{ on } \partial\Omega. \tag{108}$$

Hence, for any function $\mathbf{v}_h \in V_h$ such that $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$, we have

$$\|\mathbf{curl}\ \mathbf{u}_h^1\|_{L^2(\Omega)^3}^2 = 2E(\mathbf{u}_h^1, 0) \leqslant 2E(\mathbf{v}_h, 0) = \|\mathbf{curl}\ \mathbf{v}_h\|_{L^2(\Omega)^3}^2.$$

This completes the proof. □

Choosing $\lambda = \lambda^\dagger$ in the definition of $a_\lambda(\mathbf{u}, \mathbf{v})$, we get

$$a_{\lambda^\dagger}(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{curl}\ \mathbf{u} - \lambda^\dagger \mathbf{u}) \cdot (\mathbf{curl}\ \mathbf{v} - \lambda^\dagger \mathbf{v})\, d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in H(\mathbf{curl}; \Omega).$$

As in the proofs of Theorem 1 and Theorem 3, we can obtain the following two results.

Lemma 17. *If the problem (11) admits a solution $(\mathbf{u}^\dagger, \lambda^\dagger) \in (H(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)) \times W^{1,\infty}(\bar{\Omega})$, λ_0 satisfies $\lambda_0 \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$, then the problem: Find $(\mathbf{u}, \varphi) \in H(\mathbf{curl}; \Omega) \times \mathcal{D}$ such that*

$$\begin{cases} a_{\lambda^\dagger}(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \varphi) = 0, & \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega), \\ b(\mathbf{u}, \psi) = 0, & \forall \psi \in \mathcal{D}, \\ \mathbf{u} \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} & \textit{on } \partial\Omega, \end{cases} \tag{109}$$

admits one and only one solution $(\mathbf{u}^\dagger, 0) \in H(\mathbf{curl}; \Omega) \times \mathcal{D}$.

□

Lemma 18. *If the problem (11) admits a solution $(\mathbf{u}^\dagger, \lambda^\dagger) \in (H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)) \times W^{1,\infty}(\bar{\Omega})$, λ_0 satisfies $\lambda_0 \neq 0$ on a boundary component Σ which is a nonempty open subset of $\partial\Omega$, then the saddle-point problem: Find $(\mathbf{u}_h, \varphi_h) \in X_h \times Y_h$ such that*

$$\begin{cases} a_{\lambda^\dagger}(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \varphi_h) = 0, & \forall \mathbf{v}_h \in X_{h,0}, \\ b(\mathbf{u}_h, \psi_h) = 0, & \forall \psi_h \in Y_h, \\ \mathbf{u}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} & \text{on } \partial\Omega, \end{cases} \tag{110}$$

admits one and only one solution $(\mathbf{u}_h^\dagger, \varphi_h^\dagger) \in X_h \times Y_h$. Furthermore, there is a constant C independent of h such that

$$\|\mathbf{u}_h^\dagger - \mathbf{u}^\dagger\|_{H(\mathbf{curl};\Omega)} + \|\varphi_h^\dagger\|_{H^1(\Omega)} \leqslant C\big(\inf_{\mathbf{v}_h \in X_h} \|\mathbf{v}_h - \mathbf{u}^\dagger\|_{H(\mathbf{curl};\Omega)} + \inf_{\psi_h \in Y_h} \|\psi_h\|_{H^1(\Omega)}\big),$$

where $\mathbf{v}_h \in X_h$ satisfies that $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$.

□

Now we present the estimate of $E(\mathbf{u}_h^*, \lambda_h^*)$.

Theorem 8. *Let $(\tilde{\mathbf{u}}, \tilde{\varphi})$ denote the solution of the saddle-point problem (76). If the problem (11) admits a solution $(\mathbf{u}^\dagger, \lambda^\dagger) \in (H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)) \times W^{1,\infty}(\bar{\Omega})$, then the following inequality is valid for $\forall \mathbf{v}_h, \mathbf{w}_h \in X_h$, $\forall \psi_h \in Y_h$ and $\forall \lambda_h \in Z_h$ such that $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$, $\mathbf{w}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ and $\lambda_h = \lambda_0$ on $\partial\Omega$,*

$$\begin{aligned} E(\mathbf{u}_h^*, \lambda_h^*) \leqslant C\big(1 &+ \inf_{\lambda_h \in Z_h} \|\lambda_h - \lambda^\dagger\|_{L^\infty(\Omega)}^2\big)\big(\inf_{\mathbf{v}_h \in X_h} \|\mathbf{v}_h - \mathbf{u}^\dagger\|_{H(\mathbf{curl};\Omega)}^2 \\ &+ \inf_{\mathbf{w}_h \in X_h} \|\mathbf{w}_h - \tilde{\mathbf{u}}\|_{H(\mathbf{curl};\Omega)}^2 + \inf_{\psi_h \in Y_h} \|\psi_h - \tilde{\varphi}\|_{H^1(\Omega)}^2\big), \end{aligned} \tag{111}$$

where C is a constant independent of h.

Proof. We prove Theorem 8 for two cases respectively. In one case, $\mathbf{curl}\ \tilde{\mathbf{u}} = 0$; in the other, $\mathbf{curl}\ \tilde{\mathbf{u}}$ does not vanish.

When $\mathbf{curl}\ \tilde{\mathbf{u}} = 0$ in Ω, by the inequalities (79) and (94), we have

$$\begin{aligned} E(\mathbf{u}_h^*, \lambda_h^*) &\leqslant E(\mathbf{u}_h^1, \lambda_h^0) \\ &\leqslant C\|\mathbf{curl}\ \mathbf{u}_h^1 - \mathbf{curl}\ \tilde{\mathbf{u}}\|_{L^2(\Omega)^3}^2 \\ &\leqslant C\big(\inf_{\mathbf{w}_h \in X_h} \|\mathbf{w}_h - \tilde{\mathbf{u}}\|_{H(\mathbf{curl};\Omega)}^2 + \inf_{\psi_h \in Y_h} \|\psi_h - \tilde{\varphi}\|_{H^1(\Omega)}^2\big), \end{aligned} \tag{112}$$

where $\psi_h \in Y_h$ and $\mathbf{w}_h \in X_h$ satisfying $\mathbf{w}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$.

When $\mathbf{curl}\ \tilde{\mathbf{u}}$ does not vanish, we deduce

$$\|\tilde{\mathbf{u}}\|_{L^2(\Omega)^3} \leqslant C\|\mathbf{curl}\ \tilde{\mathbf{u}}\|_{L^2(\Omega)^3}, \tag{113}$$

where C is a constant only dependent on $\tilde{\mathbf{u}}$. Hence we get

$$\begin{aligned}\|\mathbf{u}_h^1\|^2_{L^2(\Omega)^3} &\leqslant 2(\|\tilde{\mathbf{u}}-\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}+\|\tilde{\mathbf{u}}\|^2_{L^2(\Omega)^3})\\ &\leqslant 2(\|\tilde{\mathbf{u}}-\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}+C\|\mathbf{curl}\ \tilde{\mathbf{u}}\|^2_{L^2(\Omega)^3})\\ &\leqslant C\big(\|\tilde{\mathbf{u}}-\mathbf{u}_h^1\|^2_{H(\mathbf{curl};\Omega)}+\|\mathbf{curl}\ \mathbf{u}_h^1\|^2_{L^2(\Omega)^3}\big),\end{aligned} \tag{114}$$

Together with the inequalities (79) and (114), we obtain

$$\|\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}\leqslant C\big(\|\mathbf{curl}\ \mathbf{u}_h^1\|^2_{L^2(\Omega)^3}+\|\mathbf{w}_h-\tilde{\mathbf{u}}\|^2_{H(\mathbf{curl};\Omega)}+\|\psi_h-\tilde{\varphi}\|^2_{H^1(\Omega)}\big), \tag{115}$$

where $\forall\psi_h\in Y_h$ and $\forall\mathbf{w}_h\in X_h$ satisfying $\mathbf{w}_h\times\mathbf{n}=\mathbf{u}_0\times\mathbf{n}$ on $\partial\Omega$. It follows from the inequality (94) that

$$E(\mathbf{u}_h^*,\lambda_h^*)\leqslant E(\mathbf{u}_h^1,\lambda_h^1)=\frac{1}{2}\|\mathbf{curl}\ \mathbf{u}_h^1-\lambda_h^1\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}. \tag{116}$$

From the definition of λ_h^1 (setting $n=1$ in Step 2 of **Algorithm** 5.1), we know that, for $\forall\lambda_h\in Z_h$ satisfying $\lambda_h=\lambda_0$ on $\partial\Omega$,

$$\|\mathbf{curl}\ \mathbf{u}_h^1-\lambda_h^1\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}\leqslant\|\mathbf{curl}\ \mathbf{u}_h^1-\lambda_h\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}. \tag{117}$$

This, together with the inequality (116), leads to

$$E(\mathbf{u}_h^*,\lambda_h^*)\leqslant C(\|\mathbf{curl}\ \mathbf{u}_h^1\|^2_{L^2(\Omega)^3}+2(\|\lambda^\dagger\|^2_{L^\infty(\Omega)}+\|\lambda_h-\lambda^\dagger\|^2_{L^\infty(\Omega)})\|\mathbf{u}_h^1\|^2_{L^2(\Omega)^3}). \tag{118}$$

The inequalities (107) and (49) yield, for $\forall\mathbf{u}_h\in V_h$ such that $\mathbf{u}_h\times\mathbf{n}|_{\partial\Omega}=\mathbf{u}_0\times\mathbf{n}$,

$$\|\mathbf{curl}\ \mathbf{u}_h^1\|^2_{L^2(\Omega)^3}\leqslant\|\mathbf{curl}\ \mathbf{u}_h\|^2_{L^2(\Omega)^3}\leqslant C\|\mathbf{curl}\ \mathbf{u}_h-\lambda^\dagger\mathbf{u}_h\|^2_{L^2(\Omega)^3}, \tag{119}$$

where the constant C only depends on $\lambda^\dagger$. Taking $\mathbf{u}_h=\mathbf{u}_h^\dagger$ in the inequality (119), where $\mathbf{u}_h^\dagger$ is the solution of the problem (110), we derive

$$\|\mathbf{curl}\ \mathbf{u}_h^1\|^2_{L^2(\Omega)^3}\leqslant C\|\mathbf{curl}\ \mathbf{u}_h^\dagger-\lambda^\dagger\mathbf{u}_h^\dagger\|^2_{L^2(\Omega)^3}. \tag{120}$$

It follows from (11) that

$$\begin{aligned}\|\mathbf{curl}\ \mathbf{u}_h^\dagger-\lambda^\dagger\mathbf{u}_h^\dagger\|^2_{L^2(\Omega)^3} &= \|\mathbf{curl}\ (\mathbf{u}_h^\dagger-\mathbf{u}^\dagger)-\lambda^\dagger\mathbf{u}_h^\dagger+\lambda^\dagger\mathbf{u}^\dagger\|^2_{L^2(\Omega)^3}\\ &\leqslant 2(\|\mathbf{curl}\ (\mathbf{u}_h^\dagger-\mathbf{u}^\dagger)\|^2_{L^2(\Omega)^3}+\|\lambda^\dagger(\mathbf{u}_h^\dagger-\mathbf{u}^\dagger)\|^2_{L^2(\Omega)^3})\end{aligned} \tag{121}$$

Hence we obtain

$$\|\mathbf{curl}\ \mathbf{u}_h^\dagger-\lambda^\dagger\mathbf{u}_h^\dagger\|^2_{L^2(\Omega)^3}\leqslant C\|\mathbf{u}_h^\dagger-\mathbf{u}^\dagger\|^2_{H(\mathbf{curl};\Omega)}. \tag{122}$$

According to Lemma 18, we know that there is a constant C such that, for $\forall \mathbf{v}_h \in X_h$ satisfying $\mathbf{v}_h \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n}$ on $\partial\Omega$,

$$\|\mathbf{u}_h^\dagger - \mathbf{u}^\dagger\|_{H(\mathbf{curl};\Omega)} \leqslant C\|\mathbf{v}_h - \mathbf{u}^\dagger\|_{H(\mathbf{curl};\Omega)}. \tag{123}$$

The inequality (123), combining (120) and (122), indicates

$$\|\mathbf{curl}\ \mathbf{u}_h^1\|^2_{L^2(\Omega)^3} \leqslant C\|\mathbf{v}_h - \mathbf{u}^\dagger\|^2_{H(\mathbf{curl};\Omega)}. \tag{124}$$

Together with (115), (118) and (124), we deduce that the inequality (111) is valid. □

Combining Theorem 8 and the error estimate of the interpolation operator, we deduce the following result.

Theorem 9. *Let $(\tilde{\mathbf{u}}, \tilde{\varphi})$ denote the solution of the saddle-point problem (76) and assume that the problem (11) admits a solution $(\mathbf{u}^\dagger, \lambda^\dagger)$. If $\mathbf{u}^\dagger \in H^2(\Omega)^3$, $\lambda^\dagger \in W^{1,\infty}(\bar{\Omega})$, $\tilde{\mathbf{u}} \in H^2(\Omega)^3$ and $\tilde{\varphi} \in H^2(\Omega)$, then the following inequality is valid,*

$$E(\mathbf{u}_h^*, \lambda_h^*) \leqslant Ch^2\big(1 + h\|\lambda^\dagger\|^2_{W^{1,\infty}(\bar{\Omega})}\big)\big(\|\mathbf{u}^\dagger\|^2_{H^2(\Omega)^3} + \|\tilde{\mathbf{u}}\|^2_{H^2(\Omega)^3} + \|\tilde{\varphi}\|^2_{H^2(\Omega)}\big), \tag{125}$$

where C is a constant independent of h.

□

Remark 6. *By the definition of $E(\cdot,\cdot)$, we know that $(\mathbf{u}, \lambda)$ is a solution of the Beltrami field problem if $E(\mathbf{u}, \lambda) = 0$. Theorem 9 implies that $E(\mathbf{u}_h^*, \lambda_h^*) \to 0$ when $h \to 0$. These mean that the solution $(\mathbf{u}_h^*, \lambda_h^*)$ will converge to some solution $(\mathbf{u}, \lambda)$ of the Beltrami field problem with the mesh size h decreasing. Hence, the solution $(\mathbf{u}_h^*, \lambda_h^*)$ is an approximate solution of the Beltrami field problem, and the estimate of $E(\mathbf{u}_h^*, \lambda_h^*)$ can be viewed as an error estimate of the approximate solution $(\mathbf{u}_h^*, \lambda_h^*)$.*

7 Computational results

In this section, we report some numerical results to confirm the the effectiveness of the new iterative method proposed in this paper. Note that the parameter ε in **Algorithm** 5.1 is chosen as $\varepsilon = 10^{-4}$. The numerical experiments are performed on the unit cube $\Omega = [0,1] \times [0,1] \times [0,1]$. To get a triangulation of Ω, we decompose Ω into $n \times n \times n$ hexahedra with the mesh size $h = 1/n$. Furthermore, each hexahedron is decomposed into six tetrahedra in the standard way (refer to [19]). All the tetrahedra constitute the final triangulation.

7.1 An example of nonlinear force-free fields

In this subsection, we consider the Beltrami field $\mathbf{u}$ defined by

$$\mathbf{u} = \left(\frac{\rho^{k+1}}{(1+\rho^{2k+2})^{\frac{k+2}{2k+2}}} \cdot \frac{y}{\rho}, \quad \frac{\rho^{k+1}}{(1+\rho^{2k+2})^{\frac{k+2}{2k+2}}} \cdot \frac{x}{\rho}, \quad \frac{1}{(1+\rho^{2k+2})^{\frac{k+2}{2k+2}}} \right)^T,$$

where k is any nonnegative integer and $\rho = \sqrt{x^2+y^2}$. It is easy to check that

$$\lambda = -\frac{(2+k)\rho^k}{1+\rho^{2k+2}}.$$

Without loss of generality, we only consider the case with $k = 1$. With such $\mathbf{u}$ and λ, we can get the boundary data $\mathbf{u}_0 \times \mathbf{n}$ and λ_0. Let $\mathbf{u}_h^*$ and λ_h^* be the approximations generated by **Algorithm** 5.1 with finite iteration steps. In Table 1, we list the errors of the resulting approximations and the iteration counts.

TABLE 1

The errors of the approximations and the iteration counts versus the mesh size h

h	$\|\mathbf{u}_h^* - \mathbf{u}\|_{L^2(\Omega)^3}$	$\|\lambda_h^* - \lambda\|_{L^2(\Omega)}$	$\|\mathbf{curl}\ \mathbf{u}_h^* - \lambda_h^* \mathbf{u}_h^*\|_{L^2(\Omega)^3}$	iter.
1/4	7.224D−3	2.051D−2	1.413D−1	11
1/8	2.867D−3	1.241D−2	7.156D−2	20
1/16	9.362D−4	5.182D−3	3.600D−2	30
1/32	3.830D−4	2.657D−3	1.805D−2	40

The numerical results above show that the value of the objective functional $E(\mathbf{u}_h^*, \lambda_h^*) = \frac{1}{2}\|\mathbf{curl}\ \mathbf{u}_h^* - \lambda_h^* \mathbf{u}_h^*\|^2_{L^2(\Omega)^3}$ decreases as h^2, which confirms the result given in Theorem 9. From Table 1, we also see that the convergence orders of the approximations $\mathbf{u}_h^*$ and λ_h^* on the L^2 norm are about 1.5 and 1 respectively. All these indicate that **Algorithm** 5.1 is effective.

7.2 An example of linear force-free fields

It was pointed out in [2] that the convergence of some existing numerical methods loses for a large boundary data on λ_0. Here we give a numerical example to show that the convergence of **Algorithm** 5.1 still holds even if the boundary data on λ_0 is large. In this subsection, we consider the linear force-free field $\mathbf{u}$ defined by

$$\mathbf{u} = \left(-\frac{l}{k} u_0 e^{-lz} \cos kx, \quad -(1-\frac{l^2}{k^2})^{1/2} u_0 e^{-lz} \cos kx, \quad u_0 e^{-lz} \sin kx \right)^T,$$

where u_0, l, $k \in \mathbb{R}$ and $l < k$. To make sure λ large, we choose $l = 12$, $k = 15$ and $u_0 = 15$. In this case, the Beltrami field can be written as

$$\mathbf{u} = \left(-12e^{-15z}\cos 12x, -9e^{-15z}\cos 12x, 15e^{-15z}\sin 12x\right)^T \text{ and } \lambda = 12.$$

In the existing literatures, the absolute value of λ being tested was not larger than 3. While $\lambda = 12$ is chosen in our test. With $\mathbf{u}$ and λ, we can get the boundary data $\mathbf{u}_0 \times \mathbf{n}$ and λ_0. Table 2 shows the errors of the resulting approximations and the iteration counts.

TABLE 2

The errors of the approximations and the iteration counts versus the mesh size h

h	$\\|\mathbf{u}_h^* - \mathbf{u}\\|_{L^2(\Omega)^3}$	$\\|\lambda_h^* - \lambda\\|_{L^2(\Omega)}$	$\\|\mathbf{curl}\ \mathbf{u}_h^* - \lambda_h^*\mathbf{u}_h^*\\|_{L^2(\Omega)^3}$	iter.
1/8	6.45D−1	7.63D−0	3.76D+10	22
1/12	3.97D−1	6.78D−0	2.98D+10	29
1/16	2.88D−1	5.91D−0	2.46D+10	25

According to the definitions of $\mathbf{u}$ and λ, we know that the Beltrami field $\mathbf{u}$ is highly-oscillating and the boundary data on λ_0 is large. The numerical results above indicate that **Algorithm** 5.1 still possesses the finite termination in this situation. This means that **Algorithm** 5.1 is reliable.

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