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Abstract

We propose the subdivision-based finite element method as an integration of the isogeometric analysis (IGA) framework which adopts the uniform representation for geometric modeling and finite element simulation. The finite element function space is induced from the limit form of the Catmull-Clark surface subdivision containing boundary subdivision schemes which is $C^1$ continuity everywhere. It is capable of exactly representing complex geometries with any shaped boundaries which are represented as piecewise cubic B-spline curves. It is compatible with modern Computer Aided Design (CAD) software systems. The advantage of this strategy admits quadrilateral meshes of arbitrary topology. In this work, the considered computational domains are planar geometries. We establish the approximation properties of the Catmull-Clark surface subdivision function based on the Bramble-Hilbert lemma. Numerical tests are performed through three poisson's equations with the Dirichlet boundary condition where the results corroborate the theoretical proof.

Key words: Catmull-Clark Subdivision; Isogeometric Analysis; Convergence Character; Finite Element Analysis.

1 Introduction

The finite element method (FEM) is a vital technique in solving partial differential equations (PDEs), which has been widely applied to the large-scale scientific computing and engineering. Starting from the variational principal, FEM uses piecewise low order polynomials on the subdivision meshes of the domain to approximate the solution of PDEs.
The systems of Computer Aided Design (CAD) are based on the boundary structure representation (B-rep). The incompatible mathematical representations between CAD and numerical simulation based on FEM makes interoperability of CAD and FEM very challenging. This challenge today is addressed by expensive and time-consuming human intervention.

Isogeometric Analysis (IGA) was introduced by Hughes et.al [15] in 2005 which was proposed to replace the traditional finite elements by volumetric Non-Uniform Rational B-Splines (NURBS) [9, 17] or T-splines [16, 18, 19, 21]. The concept of IGA shows great potential in developing the seamless integration in CAD and FEM which shows far more accuracy than traditional FEM. It avoids the difficulty of mesh generation. Moreover, we can use $h$-refinement by knot insertion, and $p$-refinement by order elevation to improve the simulation accuracy without changing the geometry. To support more flexible geometry representation from design, IGA has incorporated T-splines into analysis, which possess T-joints and supports local refinement. The locally refined B-splines, denoted as LR B-splines, was recently proposed in [20] as an implementation of T-splines.

Subdivision is a powerful technique in surface modeling. It provides a simple and efficient method to generate smooth surfaces with arbitrary topology structure which cannot be satisfied by Bézier and B-spline. Moreover, it is capable of recovering sharp features of surface such as creases and corners. Subdivision surfaces and functions defined on them have played a key role in computer graphics and numerical analysis. A class of piecewise smooth surface representations in [7] were introduced based on subdivision to reconstruct smooth surface from scattered data. Thin-shell finite element analysis [4] was used for describing both the geometry and associated displacement fields. The limit function representation of Loop’s subdivision for triangular meshes was combined with the diffusion model to arrive at a discretized version of the diffusion problem [1]. Mixed finite element methods based on surface subdivision technology were used to construct high-order smooth surfaces with specified boundary conditions [10, 11, 12].

Subdivision surfaces are compatible with NURBS as the standard in CAD systems which are capable of the refinability of B-spline techniques. The geometry models can be refined to arrive at a satisfactory accuracy of the numerical simulation where the subdivision schemes are simple, efficient and can be applied to meshes with arbitrary topology. However, it has not gained actual and extensive application in engineering. The principal difficulty is the exact and fast evaluation of the subdivision surfaces at arbitrary parameter values. Fortunately, there are some pioneering works about them [13, 14].

There recently have been a few works on the application of subdivision methods in IGA. Volumetric IGA based on Catmull-Clark solids was investigated in [5]. For the IGA
methods over complex physical domain, Powell-Sabin splines were used as IGA tools for advection-diffusion-reaction problems [6]. The bivariate splines in the rational Bernstein-Bezier form over the triangulation was applied in IGA [8]. A reproducing kernel triangular B-spline-based finite element method was proposed to solve PDEs [22].

Contributions. In this paper, We present a make-up approach for IGA framework where we utilize the finite element function space induced from the limit form of the Catmull-Clark surface subdivision to uniformly describe the geometric domain and perform numerical simulation on it. It is compatible with NURBS as the standard in CAD system where the boundary of geometry is modelled as piecewise cubic B-spline curves. The advantage of this strategy admits quadrilateral meshes of arbitrary topology and any-shaped boundary. The computational domains considered in this paper are planar geometries. We develop the approximation character of Catmull-Clark surface subdivision function, which provides the mathematical support for the IGA based on Extended Catmull-Clark (IGA-CC) surface subdivision. We also perform numerical calculations using three Poisson’s equations with the Dirichlet boundary condition where the numerical results are consistent with the theoretical estimates. IGA-CC surface subdivision can be naturally integrated into the framework of the standard FEA.

The paper is organized as follows: Section 2 briefly reviews the Catmull-Clark subdivision schemes including boundary rules, and Stam’s fast evaluation strategy for the Catmull-Clark subdivision surfaces. In Section 3 we give the approximation properties of Catmull-Clark subdivision function space. In Section 4 we present three numerical examples using the Poisson’s equation with the Dirichlet boundary condition and compare with their theoretical results. Section 5 shows the conclusion and future work.

2 Preliminaries

In this section, we give a brief description of the key ideas of the Catmull-Clark subdivision schemes including boundary rules [2, 3], and Stam’s fast evaluation strategy for the Catmull-Clark subdivision surfaces [13].

2.1 Catmull-Clark Subdivision Surfaces

The Catmull-Clark subdivision is a generalization of bicubic B-spline subdivision, which eliminates the rigid restriction on the topology of the control mesh. It can generate a smooth surface from a control mesh of arbitrary topology by use of iterative refinement procedure. The control vertices of the refined meshes are generated from the control
vertices of the previous step by a portfolio of weight coefficients. Finally, this sequence of meshes converges to a limit surface composed of quadrilateral surface patches.

The application of Catmull-Clark subdivision around boundaries needs some modification [2] in order to treat boundary features, such as concave/convex corners, and sharp/smooth creases. It names edges as boundary, sub-boundary and interior edges. Boundary edges lie on the boundaries and are features of the control mesh in general. Sub-boundary edges are the ones that are not boundary edges but incident to boundary vertices. The remaining ones are interior edges. The Catmull-Clark subdivision schemes including boundary rules are described as follows.

**Vertex Schemes.**

1. Interior vertex with $N$ face valence: It is updated as the combination of its previous position with weight $1 - 7/(4N)$ and the sum of all adjacent vertices with weight $3/(2N^2)$ and all the remaining 1-ring vertices with weight $1/(4N^2)$ (see Fig. 1 (a)).

2. Boundary vertex: It is updated as the sum of its own previous position with weight $3/4$ and the two adjacent boundary vertices with weight $1/8$ (see Fig. 1 (b)).

3. Corner vertex: It needs to be interpolated, meaning they are fixed.

**Edge Schemes.**

1. Sub-boundary edge: The newly added vertex on a sub-boundary edge is the combination of the boundary vertex with weight $3/4 - \gamma$ and another endpoint of this edge with weight $\gamma$ and the sum of the four wing vertices of this edge with weight $1/16$, where $\gamma = 3/8 - 1/4\cos\theta_k$, $\theta_k = \pi/k$ for a boundary vertex, and $\theta_k = \alpha/k$ for a convex corner vertex, $\theta_k = (2\pi - \alpha)/k$ for a concave corner vertex. Here $k$ denotes the face valence of the boundary vertex, $\alpha$ is the angle of the two boundary edges incident to the boundary vertex (see Fig. 1 (c)).

2. Interior edges: Use the subdivision rule for the sub-boundary edge only by choosing $\gamma = 3/8$ (see Fig. 1 (d)).

3. Boundary edge: The newly added vertex on a boundary edge is the average of its adjacent boundary vertices (see Fig. 1 (e)).

**Face Schemes.** Insert a vertex at the centroid of each face (see Fig. 1 (f)).
2.2 Evaluation of Interior Catmull-Clark Subdivision Patch

We classify the quadrilateral control mesh into interior patches, sub-boundary and boundary patches. The patches containing boundary vertices are named as boundary patches, the ones adjacent to boundary patches are called sub-boundary patches, and all others are called interior ones. Next we simply describe the evaluation method for interior cases.

A quadrilateral patch with four control vertices of valence 4 is \textit{regular}. It can be exactly represented by a uniform bicubic B-spline patch:

\[
x(\xi, \eta) = \sum_{i=1}^{16} B_i(\xi, \eta)x_i,
\]

where \((\xi, \eta)\) are the barycentric coordinates of the unit square \(\hat{T} = \{ (\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1 \}\), and the index \(i\) refers to the local sorting of 16 control vertices shown in Figure 2 (a).

A quadrilateral patch where at least one of its control vertices has a valence other than 4 is called \textit{irregular}. Stam’s fast evaluation strategy can handle its evaluation. It requires each quadrilateral has only one vertex with a valence other than 4. In this strategy the mesh needs to be subdivided repeatedly until the parameter values \((\xi, \eta)\) of interest are interior to a regular patch. Each subdivision of an irregular patch produces three regular sub-patches and one irregular sub-patch (see Fig. 2 (b) and (c)), then repeated subdivision of the irregular patch produces a sequence of regular sub-patches defined as

\[
\begin{align*}
\hat{T}_1^k &= \{ (\xi, \eta) : \xi \in [2^{-k}, 2^{-k+1}], \eta \in [0, 2^{-k}] \}, \\
\hat{T}_2^k &= \{ (\xi, \eta) : \xi \in [2^{-k}, 2^{-k+1}], \eta \in [2^{-k}, 2^{-k+1}] \}, \\
\hat{T}_3^k &= \{ (\xi, \eta) : \xi \in [0, 2^{-k}], \eta \in [2^{-k}, 2^{-k+1}] \}, \\
\end{align*}
\]

with the subdivision level \(k = \text{floor}(\min(-\log_2(\xi), -\log_2(\eta)))\). Obviously, these sub-
patches can be mapped onto the unit square $\hat{T}$ through the transform

\[\hat{t}_{1,k}(\xi, \eta) = (2^k \xi - 1, 2^k \eta), \quad (\xi, \eta) \in \hat{T}_1^k,\]
\[\hat{t}_{2,k}(\xi, \eta) = (2^k \xi - 1, 2^k \eta - 1), \quad (\xi, \eta) \in \hat{T}_2^k,\]
\[\hat{t}_{3,k}(\xi, \eta) = (2^k \xi, 2^k \eta - 1), \quad (\xi, \eta) \in \hat{T}_3^k.\]

Hence the patch is defined by its restriction to each quadrilateral

\[x(\xi, \eta)|_{T_j^k} = \sum_{i=1}^{16} B_i(\hat{t}_{j,k}(\xi, \eta))x_{i}^{j,k}, \quad j = 1, 2, 3; \quad k = 1, 2, \ldots, \]

where $x_{i}^{j,k}$ are properly chosen from the control vertices $\bar{x}_k = [x_1^k, \ldots, x_{2N+1}^k]^T$. $\bar{x}^{k+1} = \bar{A}A^k\bar{x}^0$ where $A$ and $\bar{A}$ are the subdivision matrices at corresponding subdivision steps.

Stam [13] used the Jordan canonical form $A = SJS^{-1}$ where $S$ and $J$ have explicit forms so that the computation of $A^k$ is simplified to the computation of $J^k$. It makes the cost of the computation nearly independent of $k$ and hence very efficient.

![Fig 2: (a): A regular patch with its 16 control vertices. (b): An irregular patch over the shaded quadrilateral with an extraordinary vertex labeled ”1” whose valence is 5. (c): Subdividing this irregular patch once generates 3 evaluable sub-patches.](image)

### 3 Approximation Properties of the Catmull-Clark Subdivision Function Space

The Catmull-Clark subdivision generates smooth surfaces from an initial mesh of arbitrary topology. Their control meshes are local regular except at a fixed number of extraordinary points inheriting from the initial mesh. As the subdivision algorithm proceeds, the mesh becomes increasingly regular over these regions. The Catmull-Clark subdivision surfaces inherited many of the important properties from bicubic B-splines. They have the convex hull properties, local control, and $C^2$ continuity everywhere except at the extraordinary points where a continuous tangent plane is well defined.
3.1 Finite Element Function Space

The IGA framework adopts the uniform representation for the geometric computational domain and the numerical simulation on it. In this paper, the generalized bicubic B-splines are utilized for geometrical domain modeling and the formulation of isoparametric finite elements, which can be suitable for quadrilateral meshes of arbitrary topology and any shaped boundaries.

We describe the geometric domain $\Omega$ with an initial quasi-uniform quadrilateral mesh $\Omega^{(0)}_h$. Given the initial set of control vertices $x = x^0$ on the mesh $\Omega^{(0)}_h$, the sequence of finer and finer quadrilateral meshes $\Omega^{(k)}_h, k = 1, 2, \cdots$, can be achieved where the sequence of new control vertices are defined

$$x^{n+1} = Ax^n$$

with the Catmull-Clark subdivision matrix $A$. Taking the infinite number of the process (2) yields its limit representation denoted as $M$.

We employ the function space defined by the limit of the Catmull-Clark subdivision for describing the computation domain and performing the numerical simulation to arrive at a unified discretization of our problem which is $C^1$ continuous everywhere. The input quadrilateral mesh serves as the control mesh of the Catmull-Clark subdivision. We use $M_h$ to denote the discretized representation of the limit form $M$ of the subdivision for the geometric domain $\Omega$ where the discretization parameter $h$ usually denotes the grid size. The discretized form $M_h = \bigcup_{k=1}^{K} T_\alpha$, $\hat{T}_\alpha \cap \hat{T}_\beta = \emptyset$ for $\alpha \neq \beta$, where $\hat{T}_\alpha$ is the interior of the quadrilateral patch $T_\alpha$. Each patch $T_\alpha$ can be parameterized as

$$x_\alpha : \hat{T} \rightarrow T_\alpha; \ (\xi, \eta) \mapsto x_\alpha(\xi, \eta)$$

where the unit reference square $\hat{T} = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1\}$, and $(\xi, \eta)$ are the barycentric coordinates on it. The domain of each patch $T_\alpha$ on the quadrilateralization $M_h$ can always be locally represented as an explicit bicubic B-spline according to the formula (3).

The boundaries of the geometric domain are represented as the cubic B-spline curves which are preserved as the subdivision proceeds. It means that the given boundary curves are interpolated. Therefore the Catmull-Clark subdivision elements can exactly represent geometries in the same way which fully agrees with the concept of isogeometric strategy.

3.2 Precomputing the Basis Functions

We associate each control vertex $x_i$ of the discretized form $M_h$ with a Catmull-Clark subdivision basis function $\phi_i$. The computation of these basis functions and their deriva-
tives is nontrivial because of the required two rings of neighbors around each element with arbitrary topology, and the additional individual geometric data reflected in the boundary subdivision schemes. The basis functions corresponding to the interior control vertices are zero at the boundary because the boundary rules of the Catmull-Clark subdivision do not involve the interior control vertices. We have the following uniform scheme to treat the three types of patches.

**Interior patch.** Applying Stam’s algorithm for this case (see Section 2.2);

**Sub-boundary patch.** Subdividing a sub-boundary patch once will result in four interior quadrilaterals, so it is easy to evaluate them using the above method of the interior cases;

**Boundary patch.** Subdividing a boundary patch repeatedly till its sub-patches belong to the sub-boundary case, then use the above method to evaluate them.

With the description from above, we always appeal to Stam’s fast evaluation scheme [13] which is suitable for interior patches with only one extraordinary vertex. Therefore, it is necessary to first subdivide one time each patch of the initial mesh. The evaluation of basis functions over their support elements uses general Gaussian integration which only needs a few subdivision steps in order to bring Gaussian quadrature knots into a bicubic B-spline patch. In our implementation, we need to estimate the subdivision times in advance for the parameter value \((\xi, \eta)\) of any Gaussian quadrature knot \(g_i\), then adaptively carry out the evaluation. If we consider four Gaussian integration formula for an example, the most subdivision times for the three classes of patches after one time necessary initial subdivision are shown in Table 1.

We should note that, for a given control mesh, the number of the actual control vertices is not increased although we implement one time initial Catmull-Clark subdivision for the efficient use of Stam’s fast evaluation schemes, and the number of values to be solved is also not changed. The integration over each initial patch is the sum of the values on all knots of its four sub-patches with the same number of Gauss-Legendre knots.

All basis functions and their derivatives for each patch are pre-computed and stored in a data structure before solving equations. For the case of interior patches, the valence of their four control vertices uniquely determines the associated basis functions. We merge interior patches into several categories according to the list of the valences of their control vertices, then the patches with the same type of valence list share the same set of basis functions. It greatly reduces our computation cost and storage. The remaining sub-boundary and boundary patches have their uniquely associated basis functions because their individual geometric information embodies the involved boundary subdivision rules.
Table 1: The most subdivision times for the four Gauss-Legendre knots of the three types of patches.

<table>
<thead>
<tr>
<th>Gi</th>
<th>(ξi, ηi)</th>
<th>Interior</th>
<th>Sub-boundary</th>
<th>Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td>g1</td>
<td>(0.2113249, 0.2113249)</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>g2</td>
<td>(0.2113249, 0.7886751)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>g3</td>
<td>(0.7886751, 0.2113249)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>g4</td>
<td>(0.7886751, 0.7886751)</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

The second column lists the parameter value (ξi, ηi) of the Gaussian knot gi over the unit square. The third, fourth and fifth columns respectively tell us the most subdivision times for the four Gaussian knots of the above three cases of patches.

3.3 Solvability of the Interpolation for the Limit of the Subdivision

Given an initial quadrilateral control mesh, repeated application of the Catmull-Clark subdivision produces a sequence of finer control meshes. The limit of the subdivision process generates a smooth surface. The Catmull-Clark subdivision domain converges at extraordinary control vertices. The limit position for each control vertex can be found explicitly, which is described as the following lemma (see [2] for details).

**Lemma 1** Let x0 be a vertex with valence n of the initial control mesh Ωh0. Mark its 1-ring adjacent edgepoint with subscript p and 1-ring facepoints with subscript r, then all these vertices converge to a single position

\[
\mathbf{v}_0 := \frac{n}{n + 5} \mathbf{x}_0 + \frac{4}{n(n + 5)} \sum_{j=1}^{n} \mathbf{x}_{p_j} + \frac{1}{n(n + 5)} \sum_{j=1}^{n} \mathbf{x}_{r_j},
\]

as the subdivision step goes to the infinity.

**Theorem 1** Let x be a vertex with xp, j = 1, 2, · · · , ni being the 1-ring edgepoints of the control mesh Ωh, and xr, j = 1, 2, · · · , ni being the 1-ring facepoints; vi be the i-th control vertex of the quadrilateralization Mh; fi(vi) is the i-th interpolation function value; g(xi) is the i-th control function value. The system

\[
\frac{n_i}{n_i + 5} g(x_i) + \frac{4}{n_i(n_i + 5)} \sum_{j=1}^{n_i} g(x_{p_j}) + \frac{1}{n_i(n_i + 5)} \sum_{j=1}^{n_i} g(x_{r_j}) = f(v_i), \quad i = 1, \cdots, \mu,
\]

is always solvable uniquely.
Proof. For a control vertex $x_i$, we mark the 1-ring non-adjacent edgepoints with subscript $q$. Consider the subdivision rule for the centroid of each face, we have

$$\sum_{j=1}^{n_i} x_{r_j} = \frac{n_i}{4} x_i + \frac{1}{2} \sum_{j=1}^{n_i} x_{p_j} + \frac{1}{4} \sum_{j=1}^{n_i} x_{q_j}.$$ 

It follows that the equation system (2) is equivalent to

$$\frac{4n_i + 1}{4(n_i + 5)} g(x_i) + \frac{9}{2n_i(n_i + 5)} \sum_{j=1}^{n_i} g(x_{p_j}) + \frac{1}{4n_i(n_i + 5)} \sum_{j=1}^{n_i} g(x_{q_j}) = f(v_i), \quad i = 1, \ldots, \mu. \tag{3}$$

Hence we need to show that the system of equations (3) is always solvable uniquely.

Suppose $f(v_i) = 0$, we show that the corresponding homogeneous equations of (3) has only zero solution. To simplify notation, we denote

$$l_i = \frac{1}{4n_i(n_i + 5)}, \quad g_i = g(x_i), \quad g_{p_j} = g(x_{p_j}), \quad g_{q_j} = g(x_{q_j}).$$

Rewrite the homogeneous equations of system (3) into

$$(1 - 19n_i l_i) g_i + 18l_i \sum_{j=1}^{n_i} g_{p_j} + l_i \sum_{j=1}^{n_i} g_{q_j} = 0, \quad i = 1, \ldots, \mu. \tag{4}$$

On the contrary, we assume $\{g_i\}$ be a non-zero solution of system (4), and denote

$$g_\xi = \max_j |g_j|.$$ 

We assume that $g_\xi > 0$, otherwise we can multiply $(-1)$ on both side of the equation. Then if $n_\xi \geq 5$, we have, from (3),

$$0 = (1 - 19n_\xi l_\xi) g_\xi + 18l_\xi \sum_{j=1}^{n_\xi} g_{p_j} + l_\xi \sum_{j=1}^{n_\xi} g_{q_j} \geq (1 - 19n_\xi l_\xi) g_\xi - 18l_\xi \sum_{j=1}^{n_\xi} |g_{p_j}| - l_\xi \sum_{j=1}^{n_\xi} |g_{q_j}| \geq (1 - 19n_\xi l_\xi) g_\xi - 18n_\xi l_\xi g_\xi - n_\xi l_\xi g_\xi = (1 - 38n_\xi l_\xi) g_\xi > 0,$$

which is a contradiction. Then we consider the remaining cases $n_\xi = 3, 4$ in the following, and show that a contradiction will be yielded again.

Firstly from the inequalities

$$0 = (1 - 19n_\xi l_\xi) g_\xi + 18l_\xi \sum_{j=1, j \neq 1}^{n_\xi} g_{p_j} + 18l_\xi g_{p_{k_1}} + l_\xi \sum_{j=1}^{n_\xi} g_{q_j} \geq (1 - 19n_\xi l_\xi) g_\xi - 18l_\xi \sum_{j=1, j \neq 1}^{n_\xi} |g_{p_j}| + 18l_\xi g_{p_{k_1}} - l_\xi \sum_{j=1}^{n_\xi} |g_{q_j}| = (1 - 38n_\xi l_\xi + 18l_\xi) g_\xi + 18l_\xi g_{p_{k_1}} \geq 18l_\xi g_{p_{k_1}},$$

which is a contradiction. Then we consider the remaining cases $n_\xi = 3, 4$ in the following, and show that a contradiction will be yielded again.
we have $g_{pk_l} \leq 0$ for any $l = 1, \ldots, n_\xi$. Now let $m$ be an index, such that

$$|g_{pk_m}| = \max_{1 \leq j \leq n_\xi} |g_{pk_j}|.$$  

Then from $(1 - 19n_\xi l_\xi) g_\xi + 18l_\xi \sum_{j=1}^{n_\xi} g_{pk_j} + l_\xi \sum_{j=1}^{n_\xi} g_{qk_j} = 0$, it is easy to observe that

$$g_{pk_m} \leq \alpha(\xi) g_\xi \quad \text{with} \quad \alpha(n_\xi) = \frac{1 - 20n_\xi l_\xi}{18n_\xi l_\xi}.$$  

Furthermore, we can derive that

$$18(g_{pk_{m-1}} + g_{pk_{m+1}}) + (g_{qk_m} + g_{qk_{m+1}}) \leq \beta(n_\xi) g_\xi \quad \text{with} \quad \beta(n_\xi) = \frac{1 - 38(n_\xi - 1)l_\xi}{l_\xi}.$$  

Now consider equation (4) for $i = k_m$. Using the inequalities obtained above, we have

$$0 = (1 - 19n_{km} l_{km}) g_{km} + 18l_{km} \sum_{j=1}^{n_{km}} g_{pj_j} + l_{km} \sum_{j=1}^{n_{km}} g_{qj_j}$$

$$= (1 - 19n_{km} l_{km}) g_{km} + 18l_{km} \sum_{j \neq m, m+1} g_{pj_j} + 18l_{km} (g_{pk_{m-1}} + g_{pk_{m+1}}) + l_{km} \sum_{j \neq m, m+1} g_{qj_j} + l_{km} (g_{qk_m} + g_{qk_{m+1}})$$

$$\leq \alpha(n_\xi)(1 - 19n_{km} l_{km}) g_\xi + 19(n_{km} - 2)l_{km} g_\xi + \beta(n_\xi) l_{km} g_\xi$$

$$= h(n_\xi, n_{km}) g_\xi,$$

where

$$h(n_\xi, n_{km}) = \alpha(n_\xi)(1 - 19n_{km} l_{km}) + 19(n_{km} - 2)l_{km} + \beta(n_\xi) l_{km}.$$  

For each fixed $n_\xi = 3, 4$, we have $h(n_\xi, n_{km}) < 0$ with respect to $n_{km} \geq 3$. Therefore $h(n_\xi, n_{km}) g_\xi < 0$. This is a contradiction again.

Hence, the homogeneous equations of (3) has only zero solution and the theorem is proved.

### 3.4 Interpolation Error with Catmull-Clark Subdivision Functions

Let $\hat{\Omega}$ be a rectangular parametric domain of points. We assume that $G$ is smooth invertible such that

$$\mathcal{M} = G(\hat{\Omega}), \quad \hat{\Omega} = G^{-1}(\mathcal{M}).$$  

It provides a parameterization for the limit representation $\mathcal{M}$ of the extended Catmull-Clark subdivision for the geometric domain $\Omega$. Therefore, each patch $T \in \mathcal{M}_h$ is mapped onto a unit square $\hat{T} \in \hat{\Omega}$. We associate the domain set $\hat{T} \in \hat{\Omega}$ as the support of $\hat{T}$ which is a unit of 1-ring neighbors of $\hat{T}$ on $\hat{\Omega}$. Analogously, the support $\hat{T}$ is mapped into

$$\hat{T} = G(\hat{T}),$$
where $\tilde{T}$ is the set of 1-ring neighbors of $T$ on the quadrangularization $\mathcal{M}_h$.

We consider the error estimation of finite element in the limit function space of the extended Catmull-Clark subdivision which is denoted as $S_h$. We need to introduce the usual Hilbert space $H^1(D)$ endowed with the norm $\| \cdot \|_{H^1(D)}$ and the semi-norm $| \cdot |_{H^1(D)}$ where $D \subset \mathbb{R}^2$ is a bounded open domain.

**Theorem 2** Let $s = 0, 1$. Given $T \in \mathcal{M}_h$ and its corresponding support extension $\tilde{T}$, there exists an interpolation function $\cap v \in S_h$ such that

$$|v - \cap v|_{H^s(T)} \leq Ch_{\tilde{T}}^{2-s} \sum_{j=1}^{2} |v|_{H^j(\tilde{T})}, \quad \forall v \in H^s(\tilde{T}),$$

where $h_{\tilde{T}}$ is the element size $h_{\tilde{T}} = \max\{h_T | T' \in \tilde{T}\}$.

**Proof.** Based on Theorem 1, we define a functional $F(\hat{u})$ on the parameter domain $\hat{\Omega}$

$$F(\hat{u}) = \hat{u}(v_i) - \left((1 - 5n_i l_i) \hat{u}(x_i) + 4l_i \sum_j^{n_i} \hat{u}(x_{p_j}) + l_i \sum_j^{n_i} \hat{u}(x_{r_j})\right), \quad i = 1, \cdots, \mu,$$

where $\hat{u}(v_i)$ is the $i$-th interpolation function value with $v_i = (1 - 5n_i l_i) \hat{x}_i + 4l_i \sum_j^{n_i} \hat{x}_{p_j} + l_i \sum_j^{n_i} \hat{x}_{r_j}$. $\hat{u}(x_i)$ is the control function value on vertex $\hat{x}_i$, with $x_j, j = 1, \cdots, n_i$ being the 1-ring neighbor vertices on the mesh $\hat{\Omega}$, and $l_i$ is defined as (1). Let $\hat{P}_1$ represent the set of piecewise linear polynomial functions on mesh $\hat{\Omega}$. We can achieve that $F(\hat{u}) = 0$ for $\hat{u} \in \hat{P}_1(\hat{\Omega})$. Recalling the Bramble-Hilbert lemma, there exists $\cap \hat{v}$ being the interpolant for $\hat{v}$ on the mesh $\hat{\Omega}$ such that

$$|\hat{v} - \cap \hat{v}|_{H^s(\hat{T})} \leq C|\hat{v}|_{H^2(\hat{T})}, \quad \forall \hat{v} \in H^2(\hat{T}),$$

where $C$ is a constant independent of $\hat{v}$.

By Lemma 3 in [23], for $v \in H^2(\tilde{T})$, $\hat{v} = v \circ G^{-1}$, we get

$$|v - \cap v|_{H^s(T)} \leq C|J_G|_{L^\infty(T)}^{1/2} \cdot |\nabla G|_{L^\infty(\hat{T})} \cdot |\hat{v} - \cap \hat{v}|_{H^s(\hat{T})},$$

Using (6), the above bound easily gives

$$|v - \cap v|_{H^s(T)} \leq C|J_G|_{L^\infty(T)}^{1/2} \cdot h_{\hat{T}}^{-s} \cdot \sum_{\hat{T} \cap \tilde{T} \neq 0} |\hat{v}|_{H^2(\hat{T})}.$$

By Lemma 3 in [23] again, we have

$$|\hat{v}|_{H^2(\hat{T})} \leq C|J_G^{-1}|_{L^\infty(\hat{T})}^{1/2} \left\{ |\nabla G|_{L^\infty(\hat{T})}^2 \cdot |v|_{H^2(T)} + |\nabla^2 G|_{L^\infty(\hat{T})} \cdot |v|_{H^1(T)} \right\}$$

$$\leq C|J_G^{-1}|_{L^\infty(\hat{T})}^{1/2} \cdot h_{\hat{T}}^2 \cdot \sum_{j=1}^{2} |v|_{H^j(T)}.$$
Hence, with the aid of (8) and (9), and $h_T = \max \{ h_T' | T' \in \tilde{T} \}$, we can obtain

$$|v - \nabla v|_{H^s(T)} \leq C |J_G|_{L_\infty(\tilde{T})}^{1/2} \cdot h_T^{2-s} \cdot \sum_{j=1}^2 \sum_{T' \cap \tilde{T} \neq 0} |J_G^{-1}|_{L_\infty(T')}^{1/2} |v|_{H^j(T')}$$

(10)

where we used $|J_G|_{L_\infty(\tilde{T})}^{1/2} |J_G^{-1}|_{L_\infty(T')} \leq C$. We finally get (5).

As a corollary of Theorem 2 we can have the global error estimate stated as follows.

**Theorem 3** Let $s = 0, 1$, we have

$$\sum_{T \in \mathcal{M}_h} |v - \nabla v|_{H^s(T)} \leq C \sum_{T \in \mathcal{M}_h} h_T^{2-s} \sum_{j=1}^2 |v|_{H^j(T)}, \quad \forall v \in H^2(\mathcal{M}).$$

(11)

Theorem 3 is essential for some applications of physical models. In the next section, we perform three numerical examples of the Poisson’s equations with the Dirichlet boundaries.

## 4 Applications for Poisson’s Problems

Consider the Poisson’s equation with the Dirichlet boundary condition

$$\begin{cases}
-\Delta u = f, \\
u|_{\partial \Omega} = 0,
\end{cases}$$

(1)

where the two-dimensional domain $\Omega$ is an open set with the Lipschitz continuous boundary $\partial \Omega$, $f : \Omega \rightarrow \mathbb{R}$ is a given function. Define the trial function space $\mathcal{S}_0 := \{ v | v \in H^1(\Omega), v|_{\partial \Omega} = 0 \}$, and let $v \in \mathcal{S}_0$ be a test function where $H^1$ is the usual Hilbert space. By multiplying a test function $v$ and integrating over the domain $\Omega$ on both sides of equation (1), the weak form of equation (1) is written as follows:

$$\begin{cases}
\text{Find } u \in \mathcal{S}_0 \text{ such that} \\
\iint_{\Omega} \nabla u \cdot \nabla v \, dx \, dy = \iint_{\Omega} f \cdot v \, dx \, dy, \quad \forall v \in \mathcal{S}_0.
\end{cases}$$

Let $\mathcal{M}_h$ be the discretized representation of the limit form $\mathcal{M}$ of the extended Catmull-Clark subdivision for the geometric domain $\Omega$, and the subscript $h$ indicates the maximum edge length of the mesh. The finite element function space $\mathcal{S}_0^h = \{ v | v \in \mathcal{S}_h, v|_{\partial \Omega} = 0 \}$
where $S_h$ is defined by the limit of the extended Catmull-Clark subdivision, then the finite element approximation of (1) is:

\[
\begin{cases}
\text{Find } u^h \in S^h_0 \text{ such that } \\
\int_{\Omega} \nabla u^h \cdot \nabla v dxdy = \int_{\Omega} f \cdot v dxdy, \quad \forall v \in S^h_0.
\end{cases}
\]  

(2)

We associate each control vertex $x_i$ on the mesh $M_h$ with a basis function $\phi_i$. Let \{x_1, \ldots, x_n\} be the set of interior control vertices, and \{x_{n+1}, \ldots, x_{n'}\} be the set of boundary control vertices. Then we have the basis description

\[u^h = \sum_{j=1}^{n} \phi_j u^h_j + \sum_{j=n+1}^{n'} \phi_j u^h_j \quad \text{and} \quad v = \sum_{i=1}^{n} \phi_i v_i,\]

where $u^h_j = 0$ ($j = n + 1, \ldots, n'$) is the Dirichlet boundary condition. The approximation problem (2) can be rewritten as

\[\sum_{j=1}^{n} u^h_j \int_{\Omega} \nabla \phi_j \cdot \nabla \phi_i dxdy = \int_{\Omega} f \cdot \phi_i dxdy, \quad i = 1, \ldots, n.\]

It yields a linear system $Ku = b$ where $u$ is the unknown vector. The stiffness matrix $K$ and the load vector $b$ are respectively described as

\[K = \int_{\Omega} \nabla \phi_j \cdot \nabla \phi_i dxdy \quad \text{and} \quad b = \int_{\Omega} f \cdot \phi_i dxdy.\]

The evaluation of local stiffness matrix $K$ and load vector $b$ over each patch uses an appropriate numerical integral formula. Note that the integrations on each patch are computed using 16-points Gaussian quadratures over it. It means each quadrilateral domain is firstly subdivided into four sub-quadrilaterals, and then a 4-point Gaussian quadrature formula is employed on each of the sub-quadrilaterals.

We solve three Poisson problems with the Dirichlet boundary condition. The numerical solving is operated on the limit representation of the extended Catmull-Clark subdivision domain, therefore the integration evaluation of the Gauss-Legendre knots was done on the quadrilateral mesh of limit representation of the extended Catmull-Clark subdivision domain.

The first example is an $L$-shape domain $\Omega_1$ which is defined as

\[\Omega_1 := \{(x, y)||(0 \leq x \leq 2) \& (0 \leq y \leq 2)) \setminus ((1 < x \leq 2) \& (1 < y \leq 2))\}.\]

The body force is

\[f(x, y) = 2\pi^2 \sin \pi x \sin \pi y,\]
and the exact solution is
\[ u = \sin \pi x \sin \pi y. \]

The second example is a square \( \Omega_2 \) which is defined as
\[ \Omega_2 := \{(x, y) | x^2 + y^2 \leq 1\}. \]

The body force is
\[ f(x, y) = -4, \]
and the exact solution is
\[ u(x, y) = x^2 + y^2 - 1. \]

The third example is a square with a hole \( \Omega_3 \) which is defined as
\[ \Omega_3 := \{(x, y)|(x^2 + y^2 \leq 1) \& (|x| < 0.3 \& |y| < 0.3)\}. \]

The body force is
\[ f(x, y) = -20x^2y^2 + 1.08x^2 + 1.08y^2 - 2(x^2 + y^2 - 1.0)(x^2 + y^2 - 0.18) + 0.0324, \]
and the exact solution is
\[ u(x, y) = (x^2 + y^2 - 1)(x^2 - 0.09)(y^2 - 0.09). \]

These three domains \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) are shown in Figures 1, 2 and 3 respectively. In the first column of Figures 1, 2 and 3, (a) is the initial coarse control mesh, and one time refinement is implemented from (a) to (b), (b) to (c) and (c) to (d) so that the number of quadrilateral patches on the refined meshes increases four times and their sizes approximately decrease by half. To show that the Catmull-Clark subdivision scheme does not require structured meshes and it can support the same meshes with any topological structure as the standard finite elements, the valences of the control vertices is in the range of 3 to 8. In this section, we apply the linear element method to solve the same three examples, and compare their accuracy, convergence and computational complexity.

4.1 Accuracy and Convergence

We firstly compare the accuracy between our IGA-CC subdivision and the linear element method. In Figures 1, 2 and 3, for the control meshes of four different density levels in the first column, their respective error distribution between the exact solutions \( u \) and the numerical solutions \( u^h \) are shown in the second and the third columns. The data of the second column represent the results from our IGA-CC subdivision, and the data of the
Table 2: $L^2$ Error Comparison of the Example $\Omega_1$ in Fig.1

<table>
<thead>
<tr>
<th>Vertices/Elements</th>
<th>$L^2$ error (CC)</th>
<th>convergence rate (CC)</th>
<th>$L^2$ error (Linear)</th>
<th>convergence rate (Linear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>97/75</td>
<td>4.451257e-02</td>
<td>3.766302</td>
<td>8.281625e-02</td>
<td>3.807483</td>
</tr>
<tr>
<td>340/(75 * 2^2)</td>
<td>1.181864e-02</td>
<td>3.917069</td>
<td>2.175092e-02</td>
<td>3.999121</td>
</tr>
<tr>
<td>1285/(75 * 4^2)</td>
<td>3.017215e-03</td>
<td>3.931761</td>
<td>5.438925e-03</td>
<td>4.035574</td>
</tr>
<tr>
<td>4969/(75 * 8^2)</td>
<td>7.673954e-04</td>
<td>1.347745e-03</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: $L^2$ Error Comparison of the Example $\Omega_2$ in Fig.2

<table>
<thead>
<tr>
<th>Vertices/Elements</th>
<th>$L^2$ error (CC)</th>
<th>convergence rate (CC)</th>
<th>$L^2$ error (Linear)</th>
<th>convergence rate (Linear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>121/100</td>
<td>2.298970e-02</td>
<td>4.216121</td>
<td>4.603533e-02</td>
<td>4.104039</td>
</tr>
<tr>
<td>441/(100 * 2^2)</td>
<td>5.452809e-03</td>
<td>3.991527</td>
<td>1.121708e-02</td>
<td>4.264855</td>
</tr>
<tr>
<td>1681/(100 * 4^2)</td>
<td>1.366096e-03</td>
<td>3.932270</td>
<td>2.841147e-03</td>
<td>3.969643</td>
</tr>
<tr>
<td>6561/(100 * 8^2)</td>
<td>3.474065e-04</td>
<td>7.157185e-04</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The error range for both methods is decreased with the mesh refinement procedure going on. For the same control mesh, the data show us the error span produced from the linear element method is much bigger than it from IGA-CC subdivision, and the error fluctuation from the former is also much bigger than the latter. Based on the numerical error comparison, we can observe that our IGA-CC subdivision converges faster than the linear element method.

We represent $L^2$ norm error $\|u - u^h\|_{L^2}$ for the two types of elements in Tables 2, 3 and 4. The first column shows the number of control vertices and quadrilateral patches of the control meshes. The $L^2$ norm error $\|u - u^h\|_{L^2}$ for the two types of elements are shown in the second and the forth columns. It is obvious to see that their $L^2$ norm error decreases with mesh refining process. IGA-CC subdivision becomes more accurate with the refinement procedure going on, i.e., its error is smaller than it based on the linear element method. Their convergence rate of $L^2$ norm error shown in the third and the fifth columns in this table also suggests that their convergence rate is around 1/4 which is very close to the theoretical results.

4.2 Computational Complexity

The time cost for the three examples is listed in Table 5, 6 and 7 where the data from the first to the fourth row correspond to the control meshes (a), (b), (c) and (d) of Figure
Fig 1: A L-shape $\Omega_1$. (a), (b), (c) and (d) are four control meshes where one time refinement is implemented from (a) to (b), (b) to (c) and (c) to (d). The corresponding distribution of the error $u - u^h$ resulting from our IGA-CC subdivision and the linear element method is respectively shown in (a'), (b'), (c') and (d') of the second column, and (a''), (b''), (c'') and (d'') of the third column.
Fig 2: A square $\Omega_2$. (a), (b), (c) and (d) are four control meshes where one time refinement is implemented from (a) to (b), (b) to (c) and (c) to (d). The corresponding distribution of the error $u - u^h$ resulting from our IGA-CC subdivision and the linear element method is respectively shown in (a'), (b'), (c') and (d') of the second column, and (a''), (b''), (c'') and (d'') of the third column.
Fig 3: A square with a hole $\Omega_3$. (a), (b), (c) and (d) are four control meshes where one time refinement is implemented from (a) to (b), (b) to (c) and (c) to (d). The corresponding distribution of the error $u - u^h$ resulting from our IGA-CC subdivision and the linear element method is respectively shown in (a'), (b'), (c') and (d') of the second column, and (a''), (b''), (c'') and (d'') of the third column.
Table 4: $L^2$ Error Comparison of the Example $\Omega_3$ in Fig.3

<table>
<thead>
<tr>
<th>Vertices/Elements</th>
<th>$L^2$ error (CC)</th>
<th>convergence rate (CC)</th>
<th>$L^2$ error (Linear)</th>
<th>convergence rate (Linear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>109/82</td>
<td>4.281047e-03</td>
<td>3.839163</td>
<td>6.784560e-03</td>
<td>3.644182</td>
</tr>
<tr>
<td>382/(82 + 2^2)</td>
<td>1.115099e-03</td>
<td>4.010371</td>
<td>1.861751e-03</td>
<td>3.468159</td>
</tr>
<tr>
<td>1420/(82 + 4^2)</td>
<td>2.780538e-04</td>
<td>3.866926</td>
<td>5.368124e-04</td>
<td>3.564972</td>
</tr>
<tr>
<td>5464/(82 + 8^2)</td>
<td>7.190565e-05</td>
<td>1.505797e-04</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1, 2 and 3 respectively. The number of control vertices is shown in the first column, and the second to the fourth columns give us the proportion of Class 1, Class 2 and Class 3 patches over the total patches.

The fifth and the sixth columns list the time cost (in seconds) of computing the basis functions and their derivatives because they should be pre-computed and saved in a data structure. The computation for the IGA-CC subdivision for the same control meshes is slower because it is unnecessary for us to compute the derivatives for the linear basis functions. You can find that the time cost does not increase four times after each refinement step for the extended Catmull-Clark subdivision strategy. As we mentioned earlier, most of Class 1 (interior) patches share the same set of basis functions which depend only on the valence list of their control vertices. With the mesh refinement going on, the increasing rate for the number of Class 1 (interior) patches is much faster than the other Class 2 and Class 3 patches, so that a large number of Class 1 (interior) patches are merged into the same categories which reduces our computation expense.

The seventh and the eighth columns show the total time consumption (in seconds) of solving the linear systems. The ninth and the tenth columns give us the iteration step of solving the linear systems. We can find that the the computation is slower for the linear elements because it spends more iteration steps solving the linear systems. Here we adopt the Gauss-Seidel iteration method in finally solving the linear systems where the initial values are set to be zero. We use C++ in Linux system running on a Dell PC with a 2.4GHz Q6600 Intel CPU, and the Gauss-Seidel iterative method where the threshold value of controlling the iteration-stopping is $6.0 \times 10^{-8}$.

5 Conclusions

We have developed the finite element method based on the extended Catmull-Clark surface subdivision which can be integrated into the framework of IGA scheme. This strategy shows some fine properties. It is capable of precise representation of complex geometries
Table 5: Quantitative Data of the Example $\Omega_1$ in Fig.1

<table>
<thead>
<tr>
<th>degree</th>
<th>proportion(%)</th>
<th>basis func.(s)</th>
<th>solving equa.(s)</th>
<th>iterative steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Class 1</td>
<td>Class 2</td>
<td>Class 3</td>
<td>Linear</td>
</tr>
<tr>
<td>97</td>
<td>17.33</td>
<td>32.00</td>
<td>50.67</td>
<td>0.01</td>
</tr>
<tr>
<td>340</td>
<td>44.33</td>
<td>24.00</td>
<td>26.67</td>
<td>0.03</td>
</tr>
<tr>
<td>1285</td>
<td>73.33</td>
<td>13.00</td>
<td>13.67</td>
<td>0.05</td>
</tr>
<tr>
<td>4969</td>
<td>86.33</td>
<td>6.75</td>
<td>6.92</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 6: Quantitative Data of the Example $\Omega_2$ in Fig.2

<table>
<thead>
<tr>
<th>degree</th>
<th>proportion(%)</th>
<th>basis func.(s)</th>
<th>solving equa.(s)</th>
<th>iterative steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Class 1</td>
<td>Class 2</td>
<td>Class 3</td>
<td>Linear</td>
</tr>
<tr>
<td>121</td>
<td>38.0</td>
<td>26.00</td>
<td>36.00</td>
<td>0.01</td>
</tr>
<tr>
<td>441</td>
<td>64.00</td>
<td>17.00</td>
<td>19.00</td>
<td>0.02</td>
</tr>
<tr>
<td>1681</td>
<td>81.00</td>
<td>9.25</td>
<td>9.75</td>
<td>0.05</td>
</tr>
<tr>
<td>6561</td>
<td>90.25</td>
<td>4.81</td>
<td>4.94</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 7: Quantitative Data of the Example $\Omega_3$ in Fig.3

<table>
<thead>
<tr>
<th>degree</th>
<th>proportion(%)</th>
<th>basis func.(s)</th>
<th>solving equa.(s)</th>
<th>iterative steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Class 1</td>
<td>Class 2</td>
<td>Class 3</td>
<td>Linear</td>
</tr>
<tr>
<td>109</td>
<td>0.00</td>
<td>34.15</td>
<td>65.85</td>
<td>0.01</td>
</tr>
<tr>
<td>382</td>
<td>34.14</td>
<td>32.93</td>
<td>32.93</td>
<td>0.03</td>
</tr>
<tr>
<td>1420</td>
<td>67.08</td>
<td>16.46</td>
<td>16.46</td>
<td>0.06</td>
</tr>
<tr>
<td>5464</td>
<td>83.54</td>
<td>8.23</td>
<td>8.23</td>
<td>0.20</td>
</tr>
</tbody>
</table>
with any shaped boundaries, possesses global $C^1$ smoothness, and the applicability of any topological quadrilateral meshes. We considered planar geometries as the computational models. We achieved the approximation characters based on the Bramble-Hilbert lemma. We performed three numerical tests by the Poisson equations with the Dirichlet boundary condition where the results were consistent with our theoretical proof.

In this paper, we illustrate the IGA based on the extended Catmull-Clark subdivision only using the Poisson equation as the numerical model. We are planning the new application of other physical models, such as elasticity and electromagnetics problems.

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References


