A Mortar Element Method for Coupling Natural Boundary Element Method and Finite Element Method for Unbounded Domain Problem *

Sheng Zhang and Dehao Yu

Institute of Computational Mathematics and Scientific/Engineering Computing,
Academy of Mathematics and Systems Science, Chinese Academy of Science,
Beijing, 100080, People's Republic of China

Abstract

In this paper, a mortar element method is presented for coupling natural boundary element method and finite element method for the Dirichlet exterior problem. Optimal error estimate is obtained and some numerical results are reported.

1 Introduction

In many fields of scientific and engineering computing, it is necessary to solve boundary values problems of partial differential equations over unbounded domains. For this kind of problems, the standard techniques such as the finite element method, which is effective for most problems over bounded domain, will meet some difficulties and the corresponding computing cost will be very high. As a alternative, boundary element method is considered and developed for this kind of problems and great progress has been made in this field. Also, boundary element method has its own weakness (for example, we will meet some difficulties when treating complicated bound domain problems and so on). So, it seems that the coupling method of boundary element method and finite element method which combines the advantages of boundary element method with those of finite element method may be more attractive for this kind of problems. Many authors have made contributions to the coupling method of this kind (refer to [11, 12, 14, 17, 18, 20, 21, 22, 24, 27, 32, 33, 34] and references therein) and research in this direction is of great importance both in theory and practical computation.

Generally, the coupling method of this kind can be described as follows. First, a artificial boundary is introduced which divide the unbounded domain into two subdomains – a bounded inner one and an unbounded outer one. Then we couple the finite element method which is used for the bounded inner one and boundary element method which is used for the unbounded outer one together.

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*E-mail address: szhang@lsec.cc.ac.cn, ydh@lsec.cc.ac.cn

There are many kinds of methods to archive this kind of coupling and mortar element method is one of them. Compared with other coupling method, the mortar element method appears to be attractive because meshes on different subdomains need not align across subdomain interface in this method which provide us a lot of flexibility to triangulation subdomains independently of each other and the matching of discretizations on subdomains is only enforced weakly. We refer to [5, 6, 7, 8, 16, 23, 25, 26, 28] and references therein for mortar element method and some applications for detail. The purpose of this paper is to investigate a mortar element method for this kind of coupling and to obtain the corresponding error estimate.

Another feature of this paper we want to mention is that natural boundary reduction method is used. Natural boundary reduction method, which is also known as the exact artificial boundary condition method, is suggested and developed first by K. Feng in 1980, D. Yu in 1982 and H. Han in 1985. And a very similar method, the so-called DtN method, has also been devised by J. B. Keller and D. Givoli in 1989. Compared with many other approaches of reduction, natural boundary reduction method has its own advantages(refer to [18, 32, 33] for detail).

The remainder of this paper is organized as follows: In section 2, we describe our model problem and introduce natural boundary reduction method and mortar element method. The corresponding error estimate is presented in section 3. In section 4, some numerical results are shown.

2 Model problem and discretization

We adopt the notations for Sobolev space, their norms and semi-norms as presented in [10, 15]. Let Ω be a Lipschitz bounded domain in $\mathbb{R}^2$, $\Omega^c = \mathbb{R}^2 \setminus (\Omega \cup \partial\Omega)$, $f \in L^2(\Omega^c)$ be a given compactly supported function. We consider the following model problem

$$\begin{cases} -\triangle u &= f, \quad in \ \ \Omega^c, \\ \quad u &= 0, \quad on \ \ \partial\Omega, \end{cases} \tag{2.1}$$

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r) \ , \qquad\qquad |\nabla u(x,y)| = O(1/r^2) \ ,$$

as $r = \sqrt{x^2+y^2} \to \infty$ where α is a constant.

Define

$$H^1_\triangle(\Omega^c) = \{v| \ \ \frac{v}{\sqrt{r^2+1}\ln(r^2+2)} \ , \frac{\partial v}{\partial x} \ , \frac{\partial v}{\partial y} \in L^2(\Omega^c) \ , \ v|_{\partial\Omega} = 0\}$$

and

$$a(w,v) = \int\int_{\Omega^c} \nabla w \cdot \nabla v dxdy \ , \qquad \forall w,v \in H^1_\triangle(\Omega^c) \ .$$

Then the corresponding variational form of (2.1) can be written as:
Find $u \in H^1_\triangle(\Omega^c)$ such that

$$a(u,v) = (f,v) \ , \qquad\qquad \forall v \in H^1_\triangle(\Omega^c) \ . \tag{2.2}$$

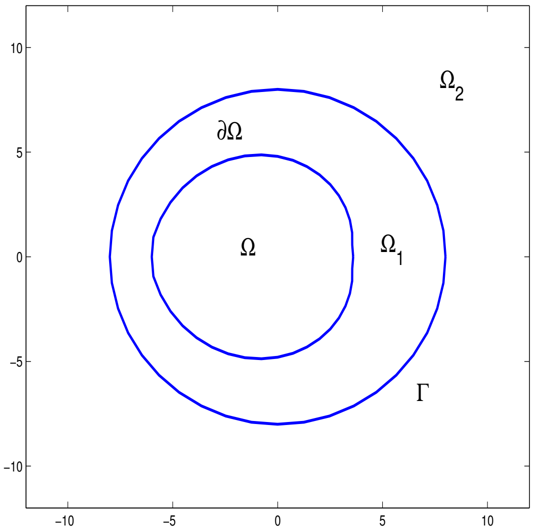

Figure 1: Artificial boundary

Due to the hypothesis on f, we can choose a circle disc Ω_0 containing $\bar{\Omega}$ and *supp* f. Let $\Omega_1 = \Omega^c \cap \Omega_0$, $\Omega_2 = \Omega_0^c = \mathbb{R}^2 \setminus (\Omega_0 \cup \partial\Omega_0)$ and $\Gamma = \partial\Omega_0$(see Figure 1). Then we have

$$a(u,v) = a_1(u,v) + a_2(u,v) \ , \tag{2.3}$$

where $a_i(u,v) = \int\int_{\Omega_i} \nabla u \cdot \nabla v dxdy \ , \quad i = 1,2.$

Next, we introduce the natural boundary reduction method.

From Green's formula on Ω_2, we have

$$a_2(u,v) = \int_\Gamma \frac{\partial}{\partial n} u(z) \cdot v(z) dz + \int\int_{\Omega_2} f v dxdy \ . \tag{2.4}$$

Let $G(z,z')$ be the Green's function for the Laplace operator on the domain Ω_2, which satisfies

$$\begin{cases} -\triangle G(z,z') &= \delta(z-z'), \quad \forall z, z' \in \Omega_2, \\ G(z,z')|_{z\in\Gamma} &= 0, \qquad\qquad \forall z' \in \Omega_2, \end{cases}$$

subject to the same asymptotic conditions as u. By taking $w = G(z,z'), v = u$ in Green's second formula

$$\int\int_{\Omega_2} (w\triangle v - v\triangle w) dz' = \int_\Gamma (w\frac{\partial v}{\partial n'} - v\frac{\partial w}{\partial n'}) dz' \ ,$$

we get (refer to [32, 33])

$$u(z) = \int\int_{\Omega_2} f(z') G(z,z') dz' - \int_\Gamma \frac{\partial}{\partial n'} G(z,z') u(z') dz' \ , \qquad \forall z \in \Omega_2 \ .$$

Thus we obtain

$$\frac{\partial u}{\partial n}(z) = \int\int_{\Omega_2} f(z') \frac{\partial}{\partial n} G(z,z') dz' - \int_\Gamma \frac{\partial^2}{\partial n \partial n'} G(z,z') u(z') dz' \ , \qquad \forall z \in \Gamma \ , \tag{2.5}$$

where n and n' denote the exterior normal vectors on Γ (viewed as the boundary of Ω_2) at the respective points z and z'.

Let

$$Ku(z) = -\int_\Gamma \frac{\partial^2}{\partial n \partial n'} G(z, z') u(z') dz' \ , \qquad\qquad z \in \Gamma \ . \tag{2.6}$$

Then, it follows from (2.4), (2.5), (2.6) and the fact that *supp* $f \subset \Omega_0$ that

$$a_2(u, v) = \int_\Gamma Ku(z) \cdot v(z) dz \ . \tag{2.7}$$

Define $H^1_*(\Omega_1) = \{v| \ \ v \in H^1(\Omega_1), \ v|_{\partial\Omega} = 0\}$ and

$$b(u, v) = a_1(u, v) + < Ku, v >_\Gamma \ , \tag{2.8}$$

where $< \cdot, \cdot >_\Gamma$ denotes the L^2 inner product on Γ. With (2.3) and (2.7), we can rewrite the variational form (2.2) as:
Find $u \in H^1_*(\Omega_1)$ such that

$$b(u, v) = \int\int_{\Omega_1} f v dx dy \ , \qquad\qquad \forall v \in H^1_*(\Omega_1) \ . \tag{2.9}$$

Remark 2.1 *The operator $K : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is just the Dirichlet-Neumann operator (Steklov-Poincaré operator) for Ω_2 (refer to [30]). So, it is symmetric and semi-positive definite with respect to the inner product $< \cdot, \cdot >_\Gamma$ (refer to [32, 33]), which indicates that $b(\cdot, \cdot)$ is symmetric, bounded and coercive in $H^1_*(\Omega_1)$. Thus, it follows from the well known Lax-Milgram Theorem that the variational problem (2.9) has unique solution $u \in H^1_*(\Omega_1)$.*

Remark 2.2 *As Γ is a circle, the Green's function $G(z, z')$ can be expressed explicitly. For example, in the case that the center of the circle Γ is the origin and its radius is R,*

$$G(z, z') = \frac{1}{4\pi} \ln \frac{R^4 + r^2 r'^2 - 2R^2 r r' \cos(\theta - \theta')}{R^2(r^2 + r'^2 - 2rr' \cos(\theta - \theta'))} \ , \qquad z = (r, \theta) \ , \quad z' = (r', \theta') \in \Omega_2 \ .$$

Moreover, we have (refer to [32, 33])

$$\frac{\partial^2}{\partial n \partial n'} G(z, z') = \frac{1}{4\pi \sin^2((\theta - \theta')/2)} \ , \qquad z = (r, \theta) \ , \quad z' = (r', \theta') \in \Gamma \ .$$

It is obvious that these explicit expressions ensure the practical use of the natural boundary reduction method. And these expressions also imply an important advantage of the natural boundary reduction method compared with many other approaches: we need not to solve any boundary integral equation associated with the unbounded subdomain and instead only calculation of certain singular integrations is needed.

Remark 2.3 *To calculate the singular integrations involved, we divide the artificial boundary* Γ *into* m *circular arcs with the same length. Let* $\{\phi_i\}_{i=1}^m$ *be the set of the nodal basis functions on* Γ. *Then we can obtain (refer to [32, 33])*

$$\begin{aligned} < K\phi_i, \phi_j >_\Gamma &= -\tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\tfrac{\phi_i(\theta)\phi_j(\theta')}{\sin^2((\theta-\theta')/2)}d\theta d\theta' \\ &= \tfrac{4m^2}{\pi^3}\sum_{k=1}^{\infty}\tfrac{1}{k^3}\sin^4\tfrac{k\pi}{m}\cos\tfrac{2k(i-j)\pi}{m} \;, \qquad i,j = 1,\cdots,m \;. \end{aligned}$$

From this expression, we can easily find that the stiffness matrix of K *is symmetric and circulant, which also allows for more efficiency and implies only small memory storage for such stiffness matrix. Moreover, since the series converges quickly, suitable short finite sum can be used to simplify the calculation.*

To derive the mortar element method, we first introduce two triangulations. For Ω_1, we choose m_1 points on Γ: $t_1^1, t_2^1, \cdots, t_{m_1}^1$. Then Ω_1 is divided into some regular quasi-uniform triangles and curved triangles (at the circle Γ) with diameter h_1 such that the nodes on Γ coincide with the m_1 points chosen. This triangulation of Ω_1 is denoted as $\mathcal{T}_1$. Furthermore, we choose another set of points on Γ : $t_1^2, t_2^2, \cdots, t_{m_2}^2$ such that Γ is divided into m_2 circular arcs with the same length h_2 which forms a triangulation denoted by $\mathcal{T}_2$. Let $e_i^1, 1 \le i \le m_1$ and $e_j^2, 1 \le j \le m_2$ denote the curved segment on Γ with two endpoints t_i^1, t_{i+1}^1 and t_j^2, t_{j+1}^2 respectively. Then two independent triangulations of Γ can be expressed as $\Gamma_{h1} = \{e_i^1\}_{i=1}^{m_1}$ and $\Gamma_{h2} = \{e_j^2\}_{j=1}^{m_2}$. For the convenience of presentation, we still need some notations.

Let

$$\begin{aligned} W_1 &= \{\ v|\ \ v \in C^0(\bar\Omega_1); \quad v|_\tau \in P_1(\tau), \ \ \forall \tau \in \mathcal{T}_1, \quad v|_{\partial\Omega} = 0\} \;, \\ U_i &= \{\ v|\ \ v \in C^0(\Gamma); \quad v|_e \in P_1(e), \ \ \forall e \in \Gamma_{hi}\} \;, \ \ i = 1,2, \\ \bar U &= \{\ v|\ \ v \in L^2(\Gamma); \quad v|_e = constant, \ \ \forall e \in \Gamma_{h2}\} \;. \end{aligned}$$

The transfer operator $T : C^0(\bar\Omega_1) \mapsto U_1$ is defined as

$$(Tv)(t_j^1) = v(t_j^1) \;, \ \ j = 1,2,\cdots,m_1, \ \ \forall v \in C^0(\bar\Omega_1) \;.$$

And the L^2-orthogonal projection operator $S : L^2(\Gamma) \mapsto \bar U$ is defined by

$$< Sw, v > = < w, v > \;, \quad \forall w \in L^2(\Gamma), \ v \in \bar U \;.$$

With these notations and operators, we can define the mortar element space as

$$W = \{\ v|\ \ v = (v_1, v_2) \in W_1 \times U_2; \quad STv_1 = Sv_2\} \;.$$

We assume that $h_2 \ge h_1$ and m_2 is an odd number. Then the mortar element method can be expressed as:
Find $u_h = (u_{h1}, u_{h2}) \in W$ such that

$$\tilde b(u_h, v) = \int\int_{\Omega_1} f v_1 dxdy \;, \qquad\qquad \forall v = (v_1, v_2) \in W \;, \tag{2.10}$$

where $\tilde b(u_h, v) = a_1(u_{h1}, v_1) + < Ku_{h2}, v_2 >_\Gamma$

Remark 2.4 *The mortar condition* $STv_1 = Sv_2$ *can be described in an explicit form:*

$$v_2(M_i) = \frac{1}{|e_i^2|}\int_{e_i^2} Tv_1 ds \ , \quad i = 1, 2, \cdots, m_2 \ ,$$

where M_i *is the midpoint of the curved segment* e_i^2*. Since* v_2 *is linear function on* e_i^2*, we have*

$$v_2(t_i^2) + v_2(t_{i+1}^2) = \frac{2}{|e_i^2|}\int_{e_i^2} Tv_1 ds \ , \quad i = 1, 2, \cdots, m_2 \ .$$

As m_2 *is an odd number, for any given* $v_1 \in W_1$*, the mortar condition determines the nodal values* $\{v_2(t_i^2)\}_{i=1}^{m_2}$ *uniquely. Also, the explicit form of the mortar condition makes the computation involved in the mortar element method easy to implement and ensures its practical use for practical computation.*

For problem (2.10), we have the following result.

Lemma 2.1 *The mortar element problem (2.10) is unisolvable.*

Proof. Since m_2 is an odd number, we can easily find that the mortar space W is a nonempty subspace of the product space $W_1 \times U_2$. Therefore, the unisolvability of the mortar element problem (2.10) follows if we can verify $\tilde{b}(v, v) = 0$ with $v = (v_1, v_2) \in W$ implies $v_1 = 0$ and $v_2 = 0$.

Now, let us do this. From $a_1(v_1, v_1) = 0$ and $v_1|_{\partial\Omega} = 0$, we know $v_1 = 0$ in Ω_1. Then it follows from the mortar condition that $v_2(t_i^2) = 0, \ i = 1, 2, \cdots, m_2$, i.e. $v_2 = 0$. This completes the proof. □

3 Error estimate

In this section, we investigate the corresponding error estimate of the mortar element method. In order to do this, some more notations are needed.

Define operator $Q_1 : H_*^1(\Omega_1) \mapsto W_1$ by

$$a_1(Q_1 v, w) = a_1(v, w) \ , \qquad \forall w \in W_1 \ .$$

For the operator Q_1, we have the following result(refer to [23]).

Lemma 3.1 *The operator* Q_1 *satisfies the following inequality*

$$\|v - Q_1 v\|_{0,\Omega_1} + h_1\|v - Q_1 v\|_{1,\Omega_1} \le Ch_1^2\|v\|_{2,\Omega_1} \ , \qquad \forall v \in H^2(\Omega_1) \cap H_*^1(\Omega_1) \ .$$

For any $v \in L^2(\Gamma)$, transfer operator $E : L^2(\Gamma) \mapsto U_2$ is defined by

$$S(Ev) = Sv \ .$$

Noting that m_2 is an odd number, we can easily find that the operator E is well defined. And by similar deduction for the explicit form of the mortar condition, we have

$$(Ev)(M_i) = \frac{1}{|e_i^2|}\int_{e_i^2} v(s) ds \ , \quad i = 1, 2, \cdots, m_2 \ .$$

Moreover, the operator E has the following properties (refer to [23]).

Lemma 3.2 *The transfer operator E satisfies*

$$\|Ev\|_{0,\Gamma} \le C\|v\|_{0,\Gamma} , \qquad \forall v \in L^2(\Gamma) ,$$
$$\|Ev\|_{1,\Gamma} \le C\|v\|_{1,\Gamma} , \qquad \forall v \in H^1(\Gamma) .$$

With Lemma 3.2, it follows from the Sobolev interpolation theory that

$$\|Ev\|_{\frac{1}{2},\Gamma} \le C\|v\|_{\frac{1}{2},\Gamma} , \qquad \forall v \in H^{\frac{1}{2}}(\Gamma) . \tag{3.1}$$

After the above preparations, we can define operator $Q : H^1_*(\Omega_1) \mapsto W$ as

$$Qv = (w_1, w_2) , \quad w_1 = Q_1 v , \quad w_2 = ETQ_1 v .$$

Noting $Sw_2 = SETQ_1v = STQ_1v = STw_1$, we find that the mortar condition is satisfied under this definition and $Qv \in W$. So, the operator Q is well defined.

For $v = (v_1, v_2) \in H^1(\Omega_1) \times H^{\frac{1}{2}}(\Gamma)$, we define $\|v\|_b^2 = \|v_1\|^2_{1,\Omega_1} + \|v_2\|^2_{\frac{1}{2},\Gamma}$. For the case of $v \in H^1(\Omega_1)$, $\|v\|_b$ can be defined in the same way by taking $v_1 = v$ and $v_2 = v|_\Gamma$.

With these notations, we first present some property of the operator Q.

Theorem 3.1 *For the operator Q, we have*

$$\|v - Qv\|_b^2 \le C(h_1^2\|v\|^2_{2,\Omega_1} + h_2^2\|v\|^2_{\frac{3}{2},\Gamma}) , \qquad \forall v \in H^2(\Omega_1) \cap H^1_*(\Omega_1) .$$

Proof. From the definition of Q and $\|\cdot\|_b$, we have

$$\|v - Qv\|_b^2 = \|v - Q_1v\|^2_{1,\Omega_1} + \|v_2 - ETQ_1v\|^2_{\frac{1}{2},\Gamma} \tag{3.2}$$

where $v_2 = v|_\Gamma$. And it follows from Lemma 3.1 that

$$\|v - Q_1v\|^2_{1,\Omega_1} \le Ch_1^2\|v\|^2_{2,\Omega_1} . \tag{3.3}$$

So, we only need to estimate the second term of the right side in (3.2).

Let $I_{h2}v_2$ be the continuous and piecewise linear interpolation function of v_2 associated with U_2, then we have

$$\|v_2 - I_{h2}v_2\|_{\frac{1}{2},\Gamma} \le Ch_2\|v_2\|_{\frac{3}{2},\Gamma} . \tag{3.4}$$

Also, it follows from the fact Tv is the continuous and piecewise linear interpolation function of v associated with U_1 that

$$\|v - Tv\|_{\frac{1}{2},\Gamma} \le Ch_1\|v\|_{\frac{3}{2},\Gamma} \le Ch_1\|v\|_{2,\Omega_1} . \tag{3.5}$$

Noticing $v_2 = v|_\Gamma$ and $EI_{h2}v_2 = I_{h2}v_2$, we obtain from (3.1), (3.4)and (3.5) that

$$\begin{aligned}
\|v_2 - ETQ_1v\|_{\frac{1}{2},\Gamma} &\le \|v_2 - I_{h2}v_2\|_{\frac{1}{2},\Gamma} + \|I_{h2}v_2 - ETv\|_{\frac{1}{2},\Gamma} \\
&\quad + \|ET(I - Q_1)v\|_{\frac{1}{2},\Gamma} \\
&\le \|v_2 - I_{h2}v_2\|_{\frac{1}{2},\Gamma} + \|E(I_{h2}v_2 - v_2)\|_{\frac{1}{2},\Gamma} \\
&\quad + \|E(v|_\Gamma - Tv)\|_{\frac{1}{2},\Gamma} + \|ET(I - Q_1)v\|_{\frac{1}{2},\Gamma} \\
&\le C(\|v_2 - I_{h2}v_2\|_{\frac{1}{2},\Gamma} + \|v - Tv\|_{\frac{1}{2},\Gamma} + \|T(I - Q_1)v\|_{\frac{1}{2},\Gamma}) \\
&\le C(h_1\|v\|_{2,\Omega_1} + h_2\|v\|_{\frac{3}{2},\Gamma} + \|T(I - Q_1)v\|_{\frac{1}{2},\Gamma}) .
\end{aligned} \tag{3.6}$$

On the other hand, the inverse inequality, Lemma 3.1, Sobolev interpolation theory and error estimates of the interpolation operator T (refer to [4, 10]) imply that, for any $\varepsilon \in (0, \frac{1}{2})$,

$$\begin{aligned} \|T(I-Q_1)v\|_{\frac{1}{2},\Gamma} &\le Ch_1^{-\frac{1}{2}}\|T(I-Q_1)v\|_{0,\Gamma} \\ &\le Ch_1^{-\frac{1}{2}}(\|(I-Q_1)v\|_{0,\Gamma} + \|(I-T)(I-Q_1)v\|_{0,\Gamma}) \\ &\le Ch_1^{-\frac{1}{2}}(\|(I-Q_1)v\|_{0,\Gamma} + h_1^{\frac{1}{2}+\varepsilon}\|(I-Q_1)v\|_{\frac{1}{2}+\varepsilon,\Gamma}) \\ &\le C(h_1^{-\frac{1}{2}}\|(I-Q_1)v\|_{0,\Gamma} + h_1^{\varepsilon}\|(I-Q_1)v\|_{1+\varepsilon,\Omega_1}) \\ &\le C(h_1^{-\frac{1}{2}}\|(I-Q_1)v\|_{0,\Gamma} + h_1\|v\|_{2,\Omega_1}) \end{aligned} \tag{3.7}$$

Also, from the trace inequality and Lemma 3.1, we have

$$\|(I-Q_1)v\|_{0,\Gamma}^2 \le C\|(I-Q_1)v\|_{0,\Omega_1}\|(I-Q_1)v\|_{1,\Omega_1} \le Ch_1^3\|v\|_{2,\Omega_1}^2 \ . \tag{3.8}$$

Thus, (3.7) and (3.8) implies

$$\|T(I-Q_1)v\|_{\frac{1}{2},\Gamma} \le Ch_1\|v\|_{2,\Omega_1} \ . \tag{3.9}$$

With (3.2), (3.3), (3.6) and (3.9), we obtain the desired result. This completes the proof. □

In order to obtain the corresponding error estimate, we first recall the second Strang Lemma (refer to [10, 15]).

Lemma 3.3 *Let u be the solution of problem (2.1) and u_h be the solution of problem (2.10), then*

$$\|u-u_h\|_b \le c(\inf_{v\in W}\|u-v\|_b + \sup_{0\ne v=(v_1,v_2)\in W}\frac{|\tilde{b}(u,v)-(f,v_1)|}{\|v\|_b}) \ . \tag{3.10}$$

The first term in (3.10) is known as the approximation error and the second term is called the consistency error. So, the error estimate can be obtained if we give out estimates for these items.

With theorem 3.1, the estimate for approximation error can be obtained as follows.

$$\inf_{v\in W}\|u-v\|_b \le \|u-Qu\|_b \le C(h_1\|u\|_{2,\Omega_1} + h_2\|u\|_{\frac{3}{2},\Gamma}) \ . \tag{3.11}$$

For the estimate of the consistency error, we note that

$$\begin{aligned} |\tilde{b}(u,v)-(f,v_1)| &= |a_1(u,v_1) + < Ku, v_2 >_\Gamma - (f,v_1)| \\ &= |\int\int_{\Omega_1}\nabla u\cdot\nabla v_1 dxdy - \int_\Gamma \frac{\partial u}{\partial n}v_2 ds - (f,v_1)| \\ &= |\int\int_{\Omega_1}(-\triangle u)v_1 dxdy + \int_\Gamma \frac{\partial u}{\partial n}[v]ds - (f,v_1)| \\ &= |\int_\Gamma \frac{\partial u}{\partial n}[v]ds| \ , \end{aligned} \tag{3.12}$$

where $[v] = v_1|_\Gamma - v_2$ denotes the jump of v on Γ. And it follows from the definition of S that

$$\int_\Gamma S\frac{\partial u}{\partial n}(Tv_1 - STv_1)ds = 0 \ .$$

Thus, by using the mortar condition, we get

$$\begin{aligned}
|\int_\Gamma \tfrac{\partial u}{\partial n}[v]ds| &= |\int_\Gamma \tfrac{\partial u}{\partial n}(v_1 - Tv_1)ds - \int_\Gamma \tfrac{\partial u}{\partial n}(v_2 - Sv_2)ds \\
&\quad + \int_\Gamma \tfrac{\partial u}{\partial n}(Tv_1 - Sv_2)ds| \\
&= |\int_\Gamma \tfrac{\partial u}{\partial n}(v_1 - Tv_1)ds - \int_\Gamma \tfrac{\partial u}{\partial n}(v_2 - Sv_2)ds \\
&\quad + \int_\Gamma \tfrac{\partial u}{\partial n}(Tv_1 - STv_1)ds| \\
&= |\int_\Gamma \tfrac{\partial u}{\partial n}(v_1 - Tv_1)ds - \int_\Gamma \tfrac{\partial u}{\partial n}(v_2 - Sv_2)ds \\
&\quad + \int_\Gamma (I-S)\tfrac{\partial u}{\partial n}(Tv_1 - STv_1)ds| \\
&\le |\int_\Gamma \tfrac{\partial u}{\partial n}(v_1 - Tv_1)ds| + |\int_\Gamma \tfrac{\partial u}{\partial n}(v_2 - Sv_2)ds| \\
&\quad + |\int_\Gamma (I-S)\tfrac{\partial u}{\partial n}(Tv_1 - STv_1)ds| \\
&\triangleq I + II + III\ .
\end{aligned} \tag{3.13}$$

Next, we estimate the above three items respectively.

For the first item I, it can be done as follows.

Noting that $v_1|_{K_i} \in P_1(K_i)$ and Tv_1 is the continuous and piecewise linear interpolation of v_1 on Γ, we have for any $e_i^1 \in \Gamma_{h1}$ that

$$\|v_1 - Tv_1\|_{0,e_i^1} \le Ch_1^2\|v_1\|_{2,e_i^1} \le Ch_1^{\frac{5}{2}}\|v_1\|_{2,\infty,e_i^1} = Ch_1^{\frac{5}{2}}\|v_1\|_{1,\infty,e_i^1} \le Ch_1^{\frac{3}{2}}\|v_1\|_{1,K_i}\ .$$

Squaring both sides and summing over all curved elements K_i yields

$$\|v_1 - Tv_1\|_{0,\Gamma}^2 \le Ch_1^3\|v_1\|_{1,\Omega_1}^2\ . \tag{3.14}$$

So, it follows from (3.14) and trace inequality that

$$\begin{aligned}
I = |\int_\Gamma \tfrac{\partial u}{\partial n}(v_1 - Tv_1)ds| &\le C\|\tfrac{\partial u}{\partial n}\|_{0,\Gamma}\|v_1 - Tv_1\|_{0,\Gamma} \\
&\le Ch_1^{\frac{3}{2}}\|\tfrac{\partial u}{\partial n}\|_{0,\Gamma}\|v_1\|_{1,\Omega_1} \\
&\le Ch_1^{\frac{3}{2}}\|u\|_{2,\Omega_1}\|v_1\|_{1,\Omega_1}\ .
\end{aligned} \tag{3.15}$$

To estimate the second item II, we recall some property of operator S (refer to [29]).

Lemma 3.4 *For the L^2 projection operator $S : L^2(\Gamma) \mapsto \bar{U}$, the following inequality holds*

$$\|v - Sv\|_{0,\Gamma} \le Ch_2^s\|v\|_{s,\Gamma}\ , \qquad \forall v \in H^s(\Gamma)\ , \quad 0 \le s \le 1\ .$$

From Lemma 3.4 and the fact that $\int_\Gamma S\frac{\partial u}{\partial n}(v_2 - Sv_2)ds = 0$, it follows

$$\begin{aligned}
II = |\int_\Gamma \tfrac{\partial u}{\partial n}(v_2 - Sv_2)ds| &= |\int_\Gamma (I-S)\tfrac{\partial u}{\partial n}(v_2 - Sv_2)ds| \\
&\le \|(I-S)\tfrac{\partial u}{\partial n}\|_{0,\Gamma}\|v_2 - Sv_2\|_{0,\Gamma} \\
&\le Ch_2\|\tfrac{\partial u}{\partial n}\|_{\frac{1}{2},\Gamma}\|v_2\|_{\frac{1}{2},\Gamma} \\
&\le Ch_2\|u\|_{\frac{3}{2},\Gamma}\|v_2\|_{\frac{1}{2},\Gamma}\ .
\end{aligned} \tag{3.16}$$

Next, we estimate the third item III.

Let $\bar{S}_1$ be the L^2-orthogonal projection operator from $L^2(\Gamma)$ onto U_1. The standard argument yields

$$\|w - \bar{S}_1 w\|_{0,\Gamma} \le Ch_1\|w\|_{1,\Gamma}\ , \qquad \forall w \in H^1(\Gamma)\ , \tag{3.17}$$

and for $s = 0, 1$,

$$\|\bar{S}_1 w\|_{s,\Gamma} \leq C\|w\|_{s,\Gamma} \ , \qquad \forall w \in H^s(\Gamma) \ ,$$

which implies

$$\|\bar{S}_1 w\|_{t,\Gamma} \leq C\|w\|_{t,\Gamma} \ , \quad 0 \leq t \leq 1 \ , \qquad \forall w \in H^t(\Gamma) \ , \tag{3.18}$$

by the Sobolev interpolation theory.

Also, the fact $\|w - \bar{S}_1 w\|_{0,\Gamma} \leq \|w\|_{0,\Gamma} + \|\bar{S}_1 w\|_{0,\Gamma} \leq C\|w\|_{0,\Gamma}$, (3.17) and the Sobolev interpolation theory yield

$$\|w - \bar{S}_1 w\|_{0,\Gamma} \leq Ch_1^t \|w\|_{t,\Gamma} \ , \quad 0 \leq t \leq 1 \ , \qquad \forall w \in H^t(\Gamma) \ . \tag{3.19}$$

With (3.18), (3.14), (3.19), it follows from the inverse inequality and the trace inequality that

$$\begin{aligned} \|Tv_1\|_{\frac{1}{2},\Gamma} &\leq \|\bar{S}_1 v_1\|_{\frac{1}{2},\Gamma} + \|Tv_1 - \bar{S}_1 v_1\|_{\frac{1}{2},\Gamma} \\ &\leq C(\|v_1\|_{\frac{1}{2},\Gamma} + h_1^{-\frac{1}{2}} \|Tv_1 - \bar{S}_1 v_1\|_{0,\Gamma}) \\ &\leq C[\|v_1\|_{\frac{1}{2},\Gamma} + h_1^{-\frac{1}{2}} (\|v_1 - \bar{S}_1 v_1\|_{0,\Gamma} + \|Tv_1 - v_1\|_{0,\Gamma})] \\ &\leq C[\|v_1\|_{\frac{1}{2},\Gamma} + h_1 \|v_1\|_{1,\Omega_1}] \\ &\leq C\|v_1\|_{1,\Omega_1} \end{aligned} \tag{3.20}$$

Thus, from Lemma 3.4 and (3.20), we have

$$\begin{aligned} III &= |\int_\Gamma (I - S)\tfrac{\partial u}{\partial n}(Tv_1 - STv_1)ds| \\ &\leq C\|(I - S)\tfrac{\partial u}{\partial n}\|_{0,\Gamma} \|Tv_1 - STv_1\|_{0,\Gamma} \\ &\leq Ch_2 \|\tfrac{\partial u}{\partial n}\|_{\frac{1}{2},\Gamma} \|Tv_1\|_{\frac{1}{2},\Gamma} \\ &\leq Ch_2 \|u\|_{\frac{3}{2},\Gamma} \|v_1\|_{1,\Omega_1} \ . \end{aligned} \tag{3.21}$$

Then, it follows from (3.12), (3.13), (3.15), (3.16), (3.21) that

$$|\tilde{b}(u, v) - (f, v_1)| \leq C[h_1^{\frac{3}{2}} \|u\|_{2,\Omega_1} \|v_1\|_{1,\Omega_1} + h_2 \|u\|_{\frac{3}{2},\Gamma} (\|v_1\|_{1,\Omega_1} + \|v_2\|_{\frac{1}{2},\Gamma})] \ . \tag{3.22}$$

So, from Lemma 3.3, (3.11) and (3.22), we obtain the following theorem.

Theorem 3.2 *Let u be the solution of problem (2.1) and u_h be the solution of problem (2.10), then the following error estimate holds*

$$\|u - u_h\|_b \leq C(h_1 \|u\|_{2,\Omega_1} + h_2 \|u\|_{\frac{3}{2},\Gamma}) \ .$$

Moreover, if the assumption $u|_\Gamma \in H^2(\Gamma)$ holds, then it follows (refer to [32, 33])

$$\|v_2 - I_{h2} v_2\|_{\frac{1}{2},\Gamma} \leq Ch_2^{3/2} \|v_2\|_{2,\Gamma} \ , \tag{3.23}$$

where $v_2 = u|_\Gamma$ and $I_{h2} v_2$ is the continuous and piecewise linear interpolation function of v_2 associated with U_2.

Also, we can get from Lemma 3.4 that

$$\|(I-S)\frac{\partial u}{\partial n}\|_{0,\Gamma} \leq Ch_2\|\frac{\partial u}{\partial n}\|_{1,\Gamma} \leq Ch_2\|u\|_{2,\Gamma} \ . \tag{3.24}$$

Using (3.24) in both (3.16) and (3.21) and replacing (3.4) with (3.23), we can obtain the following theorem by the same approach used to prove Theorem 3.2.

Theorem 3.3 *Let u be the solution of problem (2.1) and u_h be the solution of problem (2.10), if $u|_\Gamma \in H^2(\Gamma)$, then the following error estimate holds*

$$\|u-u_h\|_b \leq C(h_1\|u\|_{2,\Omega_1} + h_2^{3/2}\|u\|_{2,\Gamma}) \ .$$

Remark 3.1 *Since Γ is a circle used as artificial boundary in Ω^c, it is possible to have $u|_\Gamma \in H^2(\Gamma)$ if u is smooth enough in Ω^c. So, for smooth enough u, the condition $u|_\Gamma \in H^2(\Gamma)$ is not hard to satisfy and Theorem 3.3 works for all these cases.*

Remark 3.2 *An interesting application or byproduct of Theorem 3.3 we want to mention here is this error estimate can be used to balance the mesh sizes of different sides so as to improve computational efficiency. As we can see, if h_1 and h_2 are chosen such that h_1 and $h_2^{3/2}$ are of the same order, then it will be more efficient than many other choice of meshes sizes of different sides to get the same accuracy. Of course, many computational cost will be saved by this choice of mesh sizes. So, it is also very useful for practical computation.*

Remark 3.3 *In some cases, we may meet problems with concave angles in Ω_1. Fortunately, many techniques and methods are developed for finite element method to handle this situation successfully(for example adaptive finite element method and so on). On the other hand, this situation can also be treated by using boundary element method (refer to [32, 33]). So, with only slight modifications, the mortar element method presented here can also be used for this kind of problems.*

Remark 3.4 *For solving the discrete problem (2.10), similar D-N alternating method as that presented in [31, 33] associated with the coupling of natural boundary element method and finite element method by using matching triangulation on Γ can be constructed and used. And the convergence of this D-N alternating method can be obtained in similar way. Also, many other techniques and preconditioners are developed and constructed for solving the discrete problem of mortar element method in recent years (for example, refer to [1, 2, 3, 9, 13, 19] and references therein). These techniques and preconditioners can also provide many possible approaches for solving (2.10).*

4 Numerical results

In our numerical experiment, we consider the following model problem

$$\begin{cases} -\triangle u &= f, \quad in \ \ \Omega^c, \\ u &= 0, \quad on \ \ \partial\Omega, \end{cases} \tag{4.1}$$

Table 1: Numerical experiments

m_1	m_2	h_2/h_1	N	$\|u-u_{h1}\|_{E,\Omega_1}$	Rat1	$\|u-u_{h2}\|_{E,\Gamma}$	Rat2
66	33	2	528	5.5259e-2	–	1.4551e-2	–
130	65	2	2080	2.9372e-2	0.5315	5.6089e-3	0.3855
258	129	2	8256	1.4998e-2	0.5106	2.0505e-3	0.3656
514	257	2	32896	7.5618e-3	0.5042	7.3559e-4	0.3587
1026	513	2	131328	3.7949e-3	0.5019	2.6184e-4	0.3560

Table 2: Numerical experiments

m_1	m_2	h_2/h_1	N	$\|u-u_{h1}\|_{E,\Omega_1}$	Rat1	$\|u-u_{h2}\|_{E,\Gamma}$	Rat2
132	33	4	2112	2.9554e-2	–	3.9904e-3	–
260	65	4	8320	1.5044e-2	0.5090	1.4544e-3	0.3645
516	129	4	33024	7.5735e-3	0.5034	5.2094e-4	0.3582
1028	257	4	131584	3.7978e-3	0.5015	1.8529e-4	0.3557

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r)\ , \qquad |\nabla u(x,y)| = O(1/r^2)\ , \qquad r = \sqrt{x^2+y^2} \to \infty\ ,$$

where Ω is unit circle disc and $\alpha = 1$,

$$f = \begin{cases} \frac{4}{(x^2+y^2)^2}\ , & 1 < x^2+y^2 < \frac{9}{4}\ , \\ 0\ , & \frac{9}{4} \le x^2+y^2\ . \end{cases}$$

We take Γ as a circle with radius 2. In the following tables, u and $u_h = (u_{h1}, u_{h2})$ stand for the exact solution of the model problem and the computational solution respectively. The number of segments Γ is divided on the mortar side and the nonmortar side are denoted as m_1 and m_2. Notations h_1 and h_2 denote the corresponding mesh size of $\mathcal{T}_1$ and $\mathcal{T}_2$. N is used to denote the number of unknowns. $\|\cdot\|_{E,\Omega_1}$ is a discrete norm equivalent to the norm $\|\cdot\|_{1,\Omega_1}$ and $\|\cdot\|_{E,\Gamma}$ is a discrete norm equivalent to the norm $\|\cdot\|_{\frac{1}{2},\Gamma}$. The i-th element of column "Rat1" stands for the ratio of the i-th element and the (i-1)-th element of column "$\|u-u_{h1}\|_{E,\Omega_1}$" (For example, in Table 1, the second element of column "Rat1" 0.5315 is the ratio of the second element of column "$\|u-u_{h1}\|_{E,\Omega_1}$" $2.9372e-2$ and the first element of column "$\|u-u_{h1}\|_{E,\Omega_1}$" $5.5259e-2$, i.e. 0.5315= 2.9372e-2/5.5259e-2). By the same manner, The i-th element of column "Rat2" stands for the ratio of the i-th element and the (i-1)-th element of column "$\|u-u_{h2}\|_{E,\Gamma}$".

With the discrete norm $\|\cdot\|_{E,\Omega_1}$ and $\|\cdot\|_{E,\Gamma}$, we can define discrete norm $\|\cdot\|_E$ for $x_h = (x_{h1}, x_{h2}) \in W$ by

$$\|x_h\|_E^2 = \|x_{h1}\|_{E,\Omega_1}^2 + \|x_{h2}\|_{E,\Gamma}^2\ .$$

Table 3: Numerical experiments

m_1	m_2	h_2/h_1	N	$\|u-u_{h1}\|_{E,\Omega_1}$	Rat1	$\|u-u_{h2}\|_{E,\Gamma}$	Rat2
136	17	8	2176	2.9914e-2	–	2.8560e-3	–
264	33	8	8448	1.5136e-2	0.5060	1.0347e-3	0.3623
520	65	8	33280	7.5969e-3	0.5019	3.6949e-4	0.3571
1032	129	8	132096	3.8037e-3	0.5007	1.3122e-4	0.3551

Table 4: Numerical experiments

m_1	m_2	h_2/h_1	N	$\|u-u_{h1}\|_{E,\Omega_1}$	Rat1	$\|u-u_{h2}\|_{E,\Gamma}$	Rat2
272	17	16	8704	1.5321e-2	–	7.4053e-4	–
528	33	16	33792	7.6435e-3	0.4989	2.6287e-4	0.3550
1040	65	16	133120	3.8154e-3	0.4992	9.3074e-5	0.3541

Obviously, this discrete norm is equivalent to the norm $\|\cdot\|_b$ and

$$\|x_h\|_E \le \|x_{h1}\|_{E,\Omega_1} + \|x_{h2}\|_{E,\Gamma} \ .$$

Only for the convenience of illustration, we make some simple choice and do not use the more efficient choice of mesh sizes as mentioned in Remark 3.2. For practical computation, the more efficient choice of mesh sizes mentioned in Remark 3.2 is recommended. The numerical results are presented in Table 1, Table 2, Table 3 and Table 4.

From all these tables, we can find that when the mesh sizes decrease to the half of the former case, the value of "Rat1" is also about $\frac{1}{2}$ and the value of "Rat2" is about $(\frac{1}{2})^{3/2} \approx 0.3536$, which matches our theory well. So, mortar element method performs as what we expect and numerical result supports our theory well.

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