SPURIOUS BEHAVIOR OF A SYMPLECTIC INTEGRATOR

Jialin Hong\textsuperscript{1}, Hongyu Liu\textsuperscript{2} and Geng Sun\textsuperscript{2}
\textsuperscript{1}State Key Laboratory of Scientific and Engineering Computing,
Institute of Computational Mathematics and Scientific/Engineering Computing,
Academy of Mathematics and System Sciences (AMSS),
Chinese Academy of Sciences (CAS), P.O.Box 2719, Beijing 100080, P. R. China.
\textsuperscript{2}Institute of Mathematics, AMSS, CAS, Beijing 100080, P. R. China.

Abstract In this article we study the existence and behavior of spurious solutions of symplectic
Euler method for some Hamiltonian systems. It is shown that the symplectic integrator applied
to Hamiltonian systems, in general, doesn’t avoid spurious behavior, even spurious period two
solutions. The numerical results are presented.

Key words : Spurious solutions ; Symplectic Euler method.

AMS subject classification : 65L05, 65L20, 65P10, 65P20

1. Introduction

It is well known that many popular proposed numerical methods for nonlinear dynamical system
may admit spurious limit sets. The study on this subject was devoted by many scientists and has
been developed extensively since recent years (see\cite{1, 3-10} and references therein).

In order to understand the nature of the problem, we consider the logistic model of biological
population growth, the most often discussed model in this field

\[ \frac{du}{dt} = \alpha u(1 - u), \quad \alpha > 0 \quad (t > 0) \]
\[ u(0) = u^0 (> 0). \quad (1) \]

Equation (1) has two steady states, i.e., fixed points : \( u = 0 \) and \( u = 1 \). The fixed point \( u = 1 \) is
stable, while \( u = 0 \) is unstable. The solution of it

\[ u(t) = \frac{u^0}{u^0 + (1 - u^0)e^{-\alpha t}} \quad (2) \]

is monotone and \( u(t) \to 1 \) when \( t \to +\infty \) for every positive initial data \( u^0 \) and \( \alpha > 0 \), \( u(t) \uparrow 1 \) if
\( 0 < u^0 < 1 \), and \( u(t) \downarrow 1 \), if \( u^0 > 1 \).

The explicit Euler method for (1) is

\[ u_{n+1} = (1 + A)u_n - Au_n^2, \quad u_n = u[n\Delta t], \quad A = \alpha \Delta t. \quad (3) \]

It has been analyzed thoroughly that this numerical map gives stable asymptote \( u = 1 \) if and only
if \( 0 < A < 2 \) and \( 0 < u^0 < 1 + \frac{1}{A} \). When \( 0 < u^0 < 1 \), then the solution is monotone if and only if
\( 0 < A < 1 \). When \( u^0 > 1 \), then the solution is monotone if and only if \( 0 < A < 1 \) and \( 1 < u^0 < \frac{1}{A} \).
In other cases multi-period bifurcation or chaos may take place.

Up to now there have not been any results concerning the spurious solutions of symplectic inte-
grator applied to Hamiltonian ODEs. In this paper we make a first step towards such investigation,

\textsuperscript{1}This work is supported by the Director Innovation Foundation of ICMSEC and AMSS, the Foundation of CAS,
the NNSFC (No.19971089) and the Special Funds for Major State Basic Research Projects of China G1999032804
\textsuperscript{2}This work is supported by the Director Innovation Foundation of Institute of Mathematics and AMSS.
and this is done in part through the consideration of a model problem that is a Hamiltonian system, and is used throughout this article

\[
\begin{aligned}
\dot{p} &= \alpha p(1 - p) & p(0) &= p^0 > 0 \\
\dot{q} &= \alpha(2p - 1)q + f(p) & q(0) &= q^0 > 0, \alpha > 0,
\end{aligned}
\]

(4)

where we take \( f(p) = 0 \) and \( f(p) = \alpha p \), respectively. Obviously, it is modified from the Logistic equation. The Hamiltonian of (4) is \( H(p, q) = \alpha p(1 - p)q + F(p) \), where \( F(p) \) is a primitive function of \( f(p) \).

The paper is organized as follows. In the rest of this section, we will give some results on true numerical properties of the symplectic Euler method applied to the system (4). That is, it is devoted for the analysis of the conditions under which the numerical solutions would show physically relevant behavior, i.e., they are monotonicity and asymptote preserving. In section 2, the spurious behavior of the symplectic Euler method and the existence of spurious period two solutions are discussed. In section 3, we present some numerical results. The paper is concluded in section 4.

We firstly take \( f(p) = 0 \) in (4) and discuss the properties of its true solutions. Here it is easy to obtain that

\[
\begin{aligned}
p(t) &= \frac{p^0}{p^0 + (1 - p^0)e^{-\alpha t}}, & q(t) &= q^0[p^0 + (1 - p^0)e^{-\alpha t}]^2 e^{\alpha t}.
\end{aligned}
\]

The properties of \( p(t) \) have been discussed above, while \( q(t) \uparrow +\infty (t \to +\infty) \) if \( p^0 > \frac{1}{2} \) and if \( 0 < p^0 < \frac{1}{2} \), \( q(t) \) is first monotonically decreasing for \( t < t_0 = \frac{\ln(1 - p^0) - \ln p^0}{\ln \alpha} \) and then \( q(t) \uparrow +\infty \) for \( t(> t_0) \to +\infty \). Next, with the symplectic Euler method (see [2]) we make the discretization for it and the final scheme reads

\[
\begin{aligned}
p_{n+1} &= (1 + A)p_n - Ap_n^2, \quad (5) \\
q_{n+1} &= q_n + A(2p_n - 1)q_{n+1}, \quad (6)
\end{aligned}
\]

where

\[
p_n \approx p(n\Delta t), \quad q_n \approx q(n\Delta t), \quad h = \Delta t, \quad A = \alpha h.
\]

For the numerical scheme (5-6) to give correct physical solutions, i.e., which may preserve all important properties (asymptote, monotonicity, etc.) of the physical solutions of the original dynamical system, combining the results in [1] (see [1], Theorem 2, 3, 4), we can get the following results.

**Proposition 1** The sequence \( \{p_n\} \to 1 \) and \( \{q_n\} \to +\infty \) as \( n \to +\infty \) if and only if \( A < 1 \) and \( p^0 < \frac{1 + A}{2A} \), where \( \{p_n, q_n\} \) satisfies (5-6).

**Proof.** When \( A < 1 \) and \( p^0 < \frac{1 + A}{2A} \), by Theorem 2 in [1], it is easy to know that \( \{p_n\} \to 1(n \to +\infty) \), meanwhile, from (6) we can solve that \( \frac{q_{n+1}}{q_n} = \frac{1}{1 + A - 2Ap_n} \) and hence

\[
\lim_{n \to +\infty} \frac{q_{n+1}}{q_n} = \frac{1}{1 - A} > 1,
\]

which ensures that \( \{q_n\} \to +\infty \) as \( n \to +\infty \).

On the other hand, if \( \{p_n\} \to 1 \) and \( \{q_n\} \to +\infty \) as \( n \to +\infty \), again by Theorem 1 in [1], we
know that \( A < 2 \) and \( p^0 < 1 + \frac{1}{A} \). Now, we must have \( A < 1 \), otherwise, by Lemma 9 in [1], there exists a subsequence \( \{p_n\} \) of \( \{p_n\} \) such that \( p_n \geq 1 \) for all \( n \), and hence \( \frac{q_{n+1}}{q_n} < 0 \), contradicting to \( \{q_n\} \to +\infty (n \to +\infty) \). Next, we assume that \( \frac{1+\frac{A}{2}}{2A} < p^0 < 1 + \frac{1}{A} \), then by (5) we can verify that \( 0 < p_1 < \frac{1+\frac{A}{2}}{2A} \) and this makes that \( 0 < p_n < \frac{1+\frac{A}{2}}{2A} \) for all \( n > 1 \), so \( \frac{q_{n+1}}{q_n} = \frac{1}{1+\frac{A}{2}A} > 0 \). But \( p_1 = \frac{p^0}{1+\frac{A}{2}A} < 0 \), and hence \( p_n < 0 \) for all \( n \geq 1 \), which contradicts to

\[
\lim_{n \to +\infty} p_n = +\infty,
\]

so \( p^0 < \frac{1+\frac{A}{2}}{2A} \). The proof is completed.

Then by summarizing Theorem 1 and Theorem 3 and 4 in [1] we again obtain

**Proposition 2** For all \( n \geq 0 \), \( p_n < 1 \) and \( p_n \uparrow 1 \), \( q_n \uparrow +\infty \) as \( n(\geq N_0) \to +\infty \) if and only if \( A < 1 \) and \( 0 < p^0 < 1 \), where \( N_0 \) is an integer dependent on \( p^0 \) and it is zero for \( \frac{1}{2} < p^0 < 1 \), where \( \{p_n, q_n\} \) satisfies (5-6).

**Proposition 3** For all \( n \geq 0 \), \( p_n \geq 1 \) and \( p_n \downarrow 1 \), \( q_n \uparrow +\infty \) as \( n \to +\infty \) if and only if \( A < 1 \) and \( 1 < p^0 < \frac{1+\frac{A}{2}}{2A} \), where \( \{p_n, q_n\} \) satisfies (5-6).

### 2. Spurious Behavior of the Symplectic Euler Method

In the previous section we have discussed the conditions under which the numerical map will give correct physically relevant solutions. When some of the conditions are destroyed, spurious solutions may take place. For our purpose, the following lemma is useful.

**Lemma 1** If \( f(x) \) is a continuous function on \((a, b)\), for every \( x \in (a, b) \), \( f(x) \in (a, b) \) and \( f(x) > x \) (or \( f(x) < x \)), then for any \( x \in (a, b) \), we have

\[
\lim_{n \to +\infty} f^n(x) = b \quad \text{or} \quad \lim_{n \to +\infty} f^n(x) = a.
\]

The following result describes the spurious behavior of the symplectic integrator (5-6).

**Theorem 1** If \( p_0 \not\in [0, 1 + \frac{1}{A}] \), then the symplectic difference system (5-6) has a spurious solution \( \{p_n, q_n\} \) which satisfies

\[
\lim_{n \to +\infty} p_n = -\infty, \quad \lim_{n \to +\infty} q_n = 0.
\]

**Proof.** (5) and (6) can be rewritten as

\[
p_{n+1} = (1 + A)p_n - Ap^2_n, \quad (7)
\]
\[
q_{n+1} = \frac{q_n}{(1 + A) - 2Ap_n}. \quad (8)
\]

Let \( P_n = \frac{A}{1 + A}p_n, Q_n = q_n \). Then the above difference system reads

\[
P_{n+1} = (1 + A)P_n(1 - P_n), \quad (9)
\]
\[
Q_{n+1} = \frac{Q_n}{(1 + A)(1 - 2P_n)}. \quad (10)
\]
Since $p_0 \notin [0, 1 + \frac{1}{A}]$, $P_0 \notin [0, 1]$. If $P_0 < 0$, we have
\[(1 + A)P_0(1 - P_0) = (1 + A)P_0 - (1 + A)P_0^2 < P_0.\]
From Lemma 1, it follows that for any $P_0 < 0$,
\[\lim_{n \to +\infty} P_n = -\infty.\]
On other hand, for $P_0 > 0$, that is $P_0 > 1$, we have
\[(1 + A)P_0(1 - P_0) = (1 + A)P_0 - (1 + A)P_0^2 < 0,\]
hence, $P_1 < 0$. Since sequences $\{P_n\}_{n=0}^{+\infty}$ and $\{P_n\}_{n=1}^{+\infty}$ have the same convergence as $n \to +\infty$,
\[\lim_{n \to +\infty} P_n = -\infty.\]
Because of
\[Q_n = \frac{Q_{n-1}}{(1 + A)(1 - 2P_{n-1})} = \frac{1}{(1 + A)^n} \prod_{k=0}^{n-1} \frac{Q_0}{1 - 2P_k},\]
obviously,
\[\lim_{n \to +\infty} Q_n = 0.\]
Therefore for $p_0 \notin [0, 1 + \frac{1}{A}]$,
\[\lim_{n \to +\infty} p_n = -\infty, \quad \lim_{n \to +\infty} q_n = 0.\]
This completes the proof.

Now we investigate the spurious period two solution of (5) and the spurious oscillation solution of (6). (5) has a spurious period two solution of the form
\[p_n = a + (-1)^n b, \quad b \neq 0\] (11)
where
\[a = \frac{A + 2}{2A}, \quad b = \pm \frac{\sqrt{A^2 - 4}}{2A}\]
and (11) is a real solution iff $A > 2$. The solution is asymptotic dependent on initial data and there exist suitable values of $A$ such that (5) has a spurious period two solution iff $0 < p^0 < \frac{3 - \sqrt{5}}{4}$ or $1 < p^0 < \frac{1 + \sqrt{2}}{2}$. Such values of $A$ are given by
\[A^{(1)}(p^0) = \frac{2p^0 - 1 + \sqrt{-4p^0h + 4p^0 + 1}}{2p^0(p^0 - 1)} \quad \text{for} \quad 1 < p^0 < \frac{1 + \sqrt{2}}{2},\]
or
\[A^{(2)}(p^0) = \frac{2p^0 - 1 - \sqrt{-4p^0h + 4p^0 + 1}}{2p^0(p^0 - 1)} \quad \text{for} \quad 0 < p^0 < \frac{3 - \sqrt{5}}{4} \quad \text{or} \quad 1 < p^0 < \frac{1 + \sqrt{2}}{2}.\]
If \( 2 < A < \sqrt{6} \), it was shown in [8] that the solution (11) is linearly stable and in [1], it was further shown that it may also globally stable in practical computation. Now we turn to a look at the solution of (6) when (5) has a period-2 solution. Putting (11) into (6) and a straightforward computation leads to

\[
q_{2n} = \frac{1}{(5 - A^2)^n} q_0^0, \quad q_{2n+1} = \frac{-1}{(5 - A^2)^n(1 \pm \sqrt{A^2 - 4})} q_0^0.
\]

(12)

It can be easily verified that when \( A > 2 \), \( \{q_n\} \) is either divergent or convergent to 0, i.e., when (5) has spurious solutions, the corresponding solutions of (6) will also show incorrect physically relevant behavior. To sum up above, we have

**Theorem 2** When \( A > 2 \), there exist \( p_0 \) and \( q_0 \) such that (5) has a spurious period two solution and (6) has a spurious oscillation solution.

A numerical method for an autonomous differential system which does not admit period two solutions is said to be regular of degree 2, denoted \( R^{(2)} \). A method which is not \( R^{(2)} \) is said to be irregular of degree 2, denoted \( IR^{(2)} \) (see [4, 5, 9]). Now we show that the symplectic Euler method applied to some Hamiltonian systems is \( IR^{(2)} \). Taking \( f(p) = \alpha p \) in (4), we can find that both the numerical solutions \( \{p_n\} \) and \( \{q_n\} \) of the symplectic Euler method

\[
p_{n+1} = (1 + A)p_n - Ap_n^2, \quad q_{n+1} = q_n + A(2p_n - 1)q_{n+1} + Ap_n,
\]

(13)

(14)

may have spurious period two solutions. By a straightforward algebraic argument, it can be shown that the periodic solution of \( p_n \) has the form (11) and \( q_n \) has the following form

\[
q_n = c + (-1)^n d,
\]

(15)

where

\[
c = -\frac{1}{2}, \quad d = \pm \frac{A}{2\sqrt{A^2 - 4}}.
\]

Such periodic solutions on the grid scale must be spurious and they exist iff \( A > 2 \). In a word, the following is concluded

*When the symplectic Euler method is applied to some Hamiltonian systems, the corresponding symplectic difference systems may have spurious period two solutions.*

Thus we have,

**Theorem 3** The symplectic Euler method applied to some Hamiltonian systems may have spurious period two solutions, so it is \( IR^{(2)} \).

Considering the parameter \( \alpha \) is sufficiently large, such incorrect physically relevant behavior can be observed at step-sizes used in practical implementations.
3. Numerical Results

In this section we do some numerical experiments for spurious solutions of the symplectic Euler methods (5-6) and (13-14). In the following numerical computation,

\[ p^0 = \frac{A + 2}{2A} + \frac{\sqrt{A^2 - 4}}{2A}. \]

We firstly take \( \alpha = 16 \) and \( h = \frac{1}{\sqrt[3]{8}} \), thus \( A \approx 2.05 \), and take \( q^0 = 0.5 \).

In 10 and 20 steps of numerical simulations (see Fig 1 and Fig 2), the behavior of spurious solution of (5-6) is very clear. For example, \( \{p_n\} \) is a period two sequence. When the number of steps in the numerical computation is 100, the sequence \( \{p_n\} \) given by (5) looks like in a straight line (see Fig 3). But in the phase portrait of numerical solutions with 100 steps, we can find out the periodicity of sequence \( \{p_n\} \) of (5) and the oscillation of sequence \( \{q_n\} \) given by (6) (see Fig 4). The figure 5 is on the spurious period two solutions of the symplectic integrator (13-14) with \( \alpha = 16 \), \( h = \frac{1}{\sqrt[3]{8}} \) and

\[ q^0 = -\frac{1}{2} + \frac{A}{2\sqrt{A^2 - 4}} = 1.75017580185204, \quad A = \alpha h, \]

the number of time steps is 100. The figure 6 shows the case of \( \alpha = 16 \), \( h = 0.176875 \) and \( q^0 = 0.20671421728205 \) for (13-14) with the number of time steps 50. With same \( \alpha \), \( h \) and \( q^0 \), when the number of time steps is larger than 57, the periodicity of spurious solution of (13-14) will be broken (see Figure 7).

![Fig. 1 – The spurious solution p (stars), q (solid line) in numerical computation in 10 steps.](image)

4. Conclusion

The analysis of this paper is specific to the model problem (4), but similar results may hold when the symplectic Euler method applied to solving other Hamiltonian systems. For example, an application of the symplectic Euler method to the Hamiltonian system

\[
\begin{align*}
\dot{x} &= -x^3 & x(0) &= x^0 > 0 \\
\dot{y} &= 3x^2y & y(0) &= y^0 > 0,
\end{align*}
\]

(16)
yields

\[ x_{n+1} = x_n - hx_n^3, \quad (17) \]
\[ y_{n+1} = y_n + 3hx_n^2y_{n+1}. \quad (18) \]

It’s easy to verify that (17) may have a period two solution of the form \( x_n = (-1)^n \sqrt{2/h} \), and then the solution of (18) is \( y_n = (-1/5)^n y^0 \), i.e., the symplectic Euler method for (16) may result in the following spurious solutions

\[ x_n = (-1)^n \sqrt{\frac{2}{h}}, \quad y_n = (-\frac{1}{5})^n y^0, \quad (19) \]

while the physical solutions of it are

\[ x(t) = \frac{x^0}{\sqrt{2x^0 t + 1}}, \quad y(t) = y^0 (2x^0 t + 1)^2 \quad (20) \]

Here, obviously, the spurious solutions are also dependent on the initial data, from the first equation of (19), \( x^0 = \sqrt{2/h} \), i.e., \( h = 2/x_0^2 \). Moreover, the symplectic Euler method applied to the following Hamiltonian system

\[
\begin{cases}
\dot{x} = -x^3 & x(0) = x^0 > 0 \\
\dot{y} = 3x^2y + 4x & y(0) = y^0 > 0,
\end{cases}
\quad (21)
\]
which has a spurious period two solution
\[ x_n = (-1)^n \sqrt{\frac{2}{h}}, \quad y_n = (-1)^n \sqrt{2h}. \] (24)

Throughout this article we have discussed that the symplectic Euler method for some Hamiltonian ODEs may admit spurious behavior, even spurious period two solutions. This motivates the investigation of the stability of the symplectic integrators for Hamiltonian system. Clearly, this has not been an exhaustive study but only an introduction. There are still many aspects to be considered in this area of research and should be investigated carefully in future.
The spurious solution $p$ (solid line), $q$ (stars). The below is the phase portrait (stars) of spurious solution in numerical computation in 50 steps.

The spurious solution

The spurious phase portrait

Fig. 6 – The above is the spurious solution $p$ (solid line), $q$ (stars). The below is the phase portrait (stars) of spurious solution in numerical computation in 50 steps.

The spurious solution

The spurious phase portrait

Fig. 7 – The above is the spurious solution $p$ (solid line), $q$ (stars). The below is the phase portrait (stars) of spurious solution in numerical computation in 80 steps.

References


[6] A. Iserles and A.M. Stuart, Unified approach to spurious solutions introduced by time discret-


