Multi-Scale Method Predicting Heat Transfer Performance of Composite material with Random Distribution

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Abstract In this paper, multi-scale analysis(MSA) methods for heat transfer performance computation of composite materials with random distribution is presented. First the representation of the materials with random grains/cavities distribution and the differential equation of heat transfer for the computation of composite materials performance are briefly described. Then the two-scale analysis(TSA) expression of heat transfer behavior of materials with random distribution and the procedure of MSA computation based on TSA are given. Finally the numerical result for heat transfer parameters computation is shown. The numerical result shows that MSA is a very effective method for predicting the heat transfer performance of composite materials with random distribution.

keywords multi-scale computational method, random grains/cavities distribution , expected homogenized heat conduction coefficients.

1 Introduction

With the rapid advance of material science and technology, composite materials are of more and more importance in engineering owing to their high intension and high rigidity. Therefore it is essential to accurately predict the physical performances of these composite materials. Many methods predicting physical and mechanical performance of materials are developed in physical science, refer to [3-5] in last decade. At the same time, in mathematics, many mathematicians such as J. L. Lions, O.A. Oleinik, etc. have proposed the homogenized method to compute material performance for composite materials with periodic configuration in [6-8]. In the practice, however many composite materials are more close to random configuration such as concrete materials, porous materials as shown in Figure 1. Up to now, the papers based on MSA to predict mechanics parameters and physical parameters of composite materials with random structure are very few. So in this paper, we will give one MSA method to predict the heat transfer parameters for composite materials with random configuration.

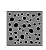
The paper remainder of this is outlined in the following way. In section 2 the representation of the materials with random grain distribution are briefly described, and the heat conduction equation with random distribution is briefly described and some results on probability fields are given in section 3. Section 4 is devoted to two-scale analysis (TSA) expression of composite materials with random configuration for evaluating the expected homogenized heat transfer coefficients of the composite materials. The procedure of MSA computation is briefly presented based on TSA in section 5. Finally, the results of numerical experiments for the heat transfer parameters of three-phase composite materials with random ellipse grains distribution is shown.

2 Representation of composite materials with random distribution

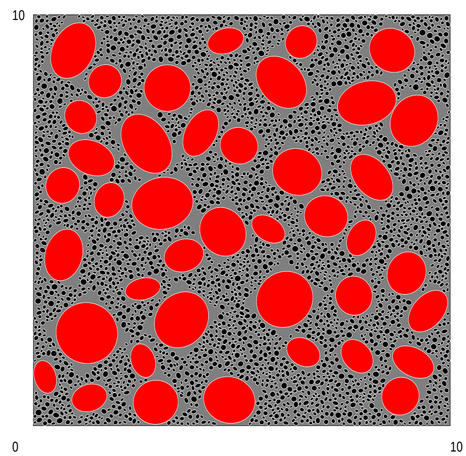
In this section, we represent composite materials with random multi-scale materials. The composite materials with random grain distribution, such as concrete and multi-phase composite materials, can be represented as follows:

All of the grains are considered as different scale ellipses in investigated structure, and then all of the ellipses are divided into several classes according to their scales(long axis). In this paper, suppose that the difference between the long axis and the short axis is not very large.

One can consider that grains inside a statistical screen for each class are subjected to certain probability distribution. In this way we suppose that there exist $\varepsilon^r(r=1,\cdots,m)$ and $\varepsilon^r >> \varepsilon^{r+1}$, the grains with the scale l^r $(\varepsilon^r > l^r)$ are subjected to certain probability distribution within ε^r statistical screen. Therefore the physical coefficients of a sample for the composite materials can be expressed as $\{k_{ij}(\frac{x}{\varepsilon^r},\omega),\omega\in P\}$, where P denotes the probability space.



a grains in ε^2-screen



b grains in ε^1-screen

Figure 1. The grains distribution model of the two screens for the three-phase composite materials

Now let us consider a structure of multi-phase composite materials, which consists of multi-scale grains. For example, the concrete material, one supposes that there are three scales of grains with the long axis's of 0.005m-0.015m, 0.015m-0.04m, 0.04m-0.08m, denoted by l^3, l^2, and l^1. They are subjected to the different probability models, respectively, shown in Figure 1. We can choose that $\varepsilon^1 = 0.2m$, $\varepsilon^2 = 0.03$m, and $\varepsilon^3 = 0.08$m. Apparently, ε^{r+1} is small scale in comparison with ε^r, and then $\varepsilon^1 > \varepsilon^2 > \varepsilon^3 > l^3 > 0$.

From the engineering survey of composite materials by the fitting method of data, the probability model P of the grains distribution inside the structure can be set up:

1. Choose the statistical screen with the size $\varepsilon^r(r=1,\cdots,m)$, which satisfy $\alpha L > \varepsilon^1 > \ldots > \varepsilon^{r+1} > \varepsilon^m > 0$, the statistical model of grains with l^r can be determined in the following way:

(a) Specify the distribution density of grains or the ratio of total volume of grains, and the distribution model of central points of them, for example, uniform distribution in the ε^r screen.

(b) Specify the distribution model of the long axis, and the middle axis and the short axis, and the distribution model of the directions for the long axis and the middle axis.

For certain size $\varepsilon^r (1 \leq r \leq m)$, from every $\omega \in P$, one sample can been obtained. For example, a sample for three-phase composite materials is shown in Figure 1. And then the periodically statistic distribution $\{k_{ij}(\frac{x}{\varepsilon^r}, \omega)\}$ on physical parameters are set up.

3 The heat transfer equation for the composite materials with periodic random structure and some results on probability space

3.1 The definition of the probability space and the related result

In this section, we shape the probability space by the 2-dimension case.

Every ellipse in two dimensions is established by five parameters: the long axis a, the short axis b, the direction θ of the long axis, the center (x_{1^0}, x_{2^0}) of the ellipses. Their probability density functions are defined as $f_a(x)$, $f_b(x)$, $f_\theta(x)$, $f_{x_{10}}(x)$, $f_{x_{20}}(x)$, respectively.

Based on the above supposition, we can obtain a samples ω^s, which is one vector,

$$\omega^s = (a_1^s, b_1^s, \theta_1^s, {x_{1^0}}_1^s, {x_{2^0}}_1^s, \ldots, a_N^s, b_N^s, \theta_N^s, {x_{1^0}}_N^s, {x_{2^0}}_N^s) \in P$$

where P denotes the probability space, N denotes the maximum numbers of grains in a screen. Therefore, for each sample the material parameters $k_{ij}(x, \omega^s)(i, j = 1, 2, \ldots, n)$ can defined:

$$k_{ij}(x, \omega^s) = \begin{cases} k_{ij}^1 & if \quad p(x) \in e_{i_1} \quad (i_1 = 1, \ldots, N) \\ k_{ij}^2 & if \quad p(x) \in \varepsilon Q^s - \cup_{i_1=1}^{i_1=N} e_{i_1} \end{cases}$$

where e_{i_1} is i_1-th ellipse and is defined:

$$\frac{(x_1 \cos\theta_{i_1}^s + x_2 \sin\theta_{i_1}^s - {x_{1^0}}_{i_1}^s \cos\theta_{i_1}^s - {x_{2^0}}_{i_1}^s \sin\theta_{i_1}^s)^2}{(a_{i_1}^s)^2} + \frac{(x_2 \cos\theta_{i_1}^s - x_1 \sin\theta_{i_1}^s + {x_{1^0}}_{i_1}^s \sin\theta_{i_1}^s - {x_{2^0}}_{i_1}^s \cos\theta_{i_1}^s)^2}{(b_{i_1}^s)^2} \leq 1$$

where k_{ij}^1, k_{ij}^2 are the constants, and $\|max\{k_{ij}^1, k_{ij}^2\}\| < M_1$, $M_1 > 0$. It is obvious that $k_{ij}(x, \omega)$ $(i, j = 1, 2, \ldots, n)$ are the random functions.

The following lemma is important to illustrate that there exists one unique expected homogenized coefficients for the composite materials with small periodic random configuration.

Theorem3.1 : If $k(\omega)$ is a bounded random function where ω is one random variable in probability space P, whose probability dense function is F(x). Then there exists a unique expectation value $\mathbf{E}_\omega k(x)$ in the probability space P.

proof: since $k(\omega)$ are bounded random function, there exists a positive constant M_1 such that

$|k(\omega)| < M_1$. One can obtain

$$\begin{aligned}\mathbf{E}_\omega k(x) &= \int_P k(\omega)\mathbf{d}\mu(\omega) = \int_{-\infty}^{+\infty} k(x)F(x)\mathbf{d}x \\ &< M_1 \int_{-\infty}^{+\infty} F(x)\mathbf{d}x \\ &< M_2\end{aligned}$$

Therefore, there exist one unique expectation value $\mathbf{E}_\omega k(x)$ in the probability space P.

3.2 The heat transfer equation for the composite materials with periodic random structure

It is well known that the heat transfer problem for the composite materials with random small periodic configuration shown in Figure 1 leads to the following equation ([see[9], [10]])

$$\begin{cases} \frac{\partial\theta^\varepsilon(x,\omega,P)}{\partial t} - \frac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon(x,\omega,P)\frac{\partial\theta^\varepsilon(x,\omega,P)}{\partial x_k}\right) = f(x) & x \in \Omega \\ \theta^\varepsilon(x,\omega,P) = \bar{\theta} & x \in \partial\Omega \\ initial \quad conditions \quad of \quad \theta^\varepsilon(x,\omega,P) \quad on \quad \partial\Omega \end{cases} \tag{1}$$

where $\theta^\varepsilon(x,\omega,P)$ denotes the increment of the temperature as compared with the reference temperature, and $k_{ij}^\varepsilon(x,\omega,P)$ represents the coefficients of the thermal conductivity, and $\bar{\theta}$ is the reference temperature on boundary, and $f(x)$ is the internal heat source, and P is the probability space.

It is supposed that the domain Ω is only composed of ε-cell with random distributions

$$\Omega = \bigcup_{t\in T} \varepsilon(Q^s + t)$$

where εQ^s denotes a cell configuration in Ω and Q^s is 1-square. Let $\{k_{ij}(\xi,\omega)\}$ be the tensor of heat transfer parameters as the above.

In order to obtain heat transfer parameters of composite materials with small periodic random configuration, we only consider the statonary problem in (1), that is assumed that

$$\frac{\partial\theta^\varepsilon(x,\omega,P)}{\partial t} = 0$$

Thus the equation (1) is changed into

$$\begin{cases} \frac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon(x,\omega,P)\frac{\partial\theta^\varepsilon(x,P)}{\partial x_j}\right) = f(x) & in \quad \Omega \\ \theta^\varepsilon(x,P) = \bar{\theta}(x) & on \quad \partial\Omega \end{cases} \tag{2}$$

Here, suppose that the probability space P and the computer simulation for the composite materials with small random periodic configuration have been performed. In common case, the tenser of the heat transfer coefficients in composite materials with small periodic random configuration satisfy the following lemma.

lemma3.2: if $k_{ij}^{\varepsilon}(x,\omega)$ are bounded, measurable and symmetric, there exist are two positive constants c_1 and c_2 (not depending on ω and ε) such that

$$c_1\eta_i\eta_i \leq k_{ij}^{\varepsilon}(x,\omega)\eta_i\eta_j \leq c_2\eta_i\eta_i, \qquad \forall x\in\Omega, \forall\omega\in P \tag{3}$$

If the coefficients in the equations (2) satisfy the condition in lemma 3.2, and $f(x)\in C^{\infty}(\Omega)$. Then the equation (2) for any $\omega\in P$ has a unique solution.

4 The two-scale formulation in probability space

In this section, we will give the expected homogenized coefficients. In the problem (2), the increment of temperature are effected not only by macroscopic behaviors but also by periodically random distribution model, hence the increment of temperature can be denoted by a function $\theta^{\varepsilon}(x,\omega,P) = \theta^{\varepsilon}(x,\xi,\omega,P)$, where the vector x represents the heat conduction behaviors of the distribution structure, denotes local coordinates in a ε-cell and the effects of periodically random distribution model and P is the probability space representing random distribution characteristic.

In order to obtain the formula of the two-scale analysis for the composite materials of random distribution with small periodicity, It is assumed that $\theta^{\varepsilon}(x,\omega,P)$ can be expanded into a series in the following form:

$$\begin{aligned}\theta^{\varepsilon}(x,\omega,P) &= \theta^0(x) + \varepsilon N_{\alpha_1}^{s}(\xi,\omega^s)W_{\alpha_1}(x) + \varepsilon^2 N_{\alpha_1\alpha_2}^{s}(\xi,\omega^s)W_{\alpha_1\alpha_2}(x)\\ &+ \varepsilon^3 P_1(x,\xi,P) \quad x\in\varepsilon Q^s\in\Omega, \quad \xi = \frac{x}{\varepsilon} - [\frac{x}{\varepsilon}]\in Q^s\end{aligned} \tag{4}$$

where Q^s is the 1-square with the configuration similar to cell $\varepsilon Q^s(\omega)$, $N_{\alpha_1}^{s}(\xi,\omega^s)$, $N_{\alpha_1\alpha_2}^{s}(\xi,\omega^s)$, $W_{\alpha_1}(x)$, $W_{\alpha_1\alpha_2}(x)$, $(\alpha_1,\alpha_2 = 1,2,\cdots n)$,are the random functions, ω^s are the samples of the random variable ω in the probability space P. Due to $\xi = \frac{x}{\varepsilon} - [\frac{x}{\varepsilon}]$, taking into account that

$$\frac{\partial\theta^{\varepsilon}(x,\omega^s,P)}{\partial x_j} = \frac{\partial\theta(x,\xi,\omega^s,P)}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial\theta(x,\xi,\omega^s,P)}{\partial\xi_j} \tag{5}$$

one obtains that:

$$\begin{aligned}k_{ij}(\xi,\omega,P)\left(\frac{\partial\theta^{\varepsilon}(x,\omega,P)}{\partial x_j}\right) &= k_{ij}(\xi,\omega^s)\frac{\partial\theta^0(x)}{\partial x_j} + k_{ij}(\xi,\omega^s)\frac{\partial N_{\alpha_1}^{s}(\xi,\omega^s)}{\partial\xi_j}W_{\alpha_1}(x)\\ &+ \varepsilon k_{ij}(\xi,\omega^s)N_{\alpha_1}^{s}(\xi,\omega^s)\frac{\partial W_{\alpha_1}(x)}{\partial x_j} + \varepsilon k_{ij}(\xi,\omega^s)\frac{\partial N_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial\xi_j}W_{\alpha_1\alpha_2}(x)\\ &+ \varepsilon^2 k_{ij}(\xi,\omega^s)N_{\alpha_1\alpha_2}^{s}(\xi,\omega^s)\frac{\partial W_{\alpha_1\alpha_2}(x)}{\partial x_j} + \varepsilon^3 T(x,\xi,\omega,P)\end{aligned} \tag{6}$$

where $T(x,\xi,\omega,P) = k_{ij}(\xi,\omega,P)\frac{\partial P_1(x,\xi,\omega,P)}{\partial x_j}$

From (6) and (2), it leads to the following equations:

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\left(k_{ij}(\xi,\omega,P)\frac{\partial \theta^\varepsilon(x,\omega,P)}{\partial x_j}\right)=\varepsilon^{-1}\left[\frac{\partial k_{ij}(\xi,\omega^s)}{\partial \xi_i}\frac{\partial \theta^0(x)}{\partial x_j}+\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\right)\right]W_{\alpha_1}(x)\\
&+k_{ij}(\xi,\omega^s)\frac{\partial^2\theta^0(x)}{\partial x_i\partial x_j}+k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\frac{\partial W_{\alpha_1}(x)}{\partial x_i}+\frac{\partial}{\partial \xi_i}(k_{ij}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s))\frac{\partial W_{\alpha_1}(x)}{\partial x_j})\\
&+\varepsilon k_{ij}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\frac{\partial^2 W_{\alpha_1}(x)}{\partial x_i\partial x_j}+\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\right)W_{\alpha_1\alpha_2}(x)\\
&+\varepsilon k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\frac{W_{\alpha_1\alpha_2}(x)}{\partial x_i}+\varepsilon\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)N^s_{\alpha_1\alpha_2}(\xi,\omega^s)\right)\frac{\partial W_{\alpha_1\alpha_2}(x)}{\partial x_j}\\
&+\varepsilon^2 k_{ij}(\xi,\omega^s)N^s_{\alpha_1\alpha_2}(\xi,\omega^s)\frac{\partial^2 W_{\alpha_1\alpha_2}(x)}{\partial x_j\partial x_i}+\varepsilon^3 V(x,\xi,\omega,P)=f(x)
\end{aligned}\tag{7}$$

where $\varepsilon^3 V(x,\xi,\omega,P)=\varepsilon^3\frac{\partial T(x,\xi,\omega,P)}{\partial x_j}$.

Since the equation (7) holds for any $\varepsilon>0$, comparing the coefficients of ε^{-1} in the equation (7), we can obtain the following equations:

$$\frac{\partial k_{ij}(\xi,\omega^s)}{\partial \xi_i}\frac{\partial \theta^0(x)}{\partial x_j}+\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\right)W_{\alpha_1}(x)=0 \tag{8}$$

Setting $W_{\alpha_1}(x)=\dfrac{\partial \theta^0(x)}{\partial x_{\alpha_1}}$, and making use of the symmetry of $\{k^\varepsilon_{ij}(\xi,\omega)\}$, by virtue of (8), one obtains that:

$$\frac{\partial k_{i\alpha_1}(\xi,\omega^s)}{\partial \xi_i}+\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\right)\frac{\partial \theta^0(x)}{\partial x_{\alpha_1}}=0 \tag{9}$$

Since $\dfrac{\partial \theta^0(x)}{\partial x_{\alpha_1}}$ does not always equal to zero in $\varepsilon Q^s\subset\Omega$. Then from (9),the following equation holds

$$\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\right)=-\frac{\partial k_{i\alpha_1}(\xi,\omega^s)}{\partial \xi_i}\qquad \xi\in Q^s \tag{10}$$

where α_1=1, $\cdots,n$.

In order to uniquely define $N^s_{\alpha_1}(\xi,\omega^s)$ onto Q^s , it is necessary to prescribe the boundary conditions. Here, choose such a following boundary condition on ∂Q^s, in order to satisfy the boundary condition in (2) for $\theta^\varepsilon(x,\omega,P)$. In Q^s, let

$$N^s_{\alpha_1}(\xi,\omega^s)=0,\qquad \xi\in\partial Q^s$$

Thus, for $\forall\omega^s\in P$, in each Q^s one obtain $N^s_{\alpha_1}(\xi,\omega^s)$ from the following equations.

$$\begin{cases}\dfrac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\dfrac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\right)=-\dfrac{\partial k_{j\alpha_1}(\xi,\omega^s)}{\partial \xi_j} & \xi\in Q^s\\ N^s_{\alpha_1}(\xi,\omega^s)=0 & \xi\in\partial Q^s\end{cases}\tag{11}$$

Based on Lax-Millgram Lemma and Poincare's inequality, it is easy to see that above equation has one unique solution.

Since the right of the equation (7) is independent of ε , we can set that the coefficients of ε^0 in the left of the equation equal to $f(x)$, i.e

$$
\begin{aligned}
& k_{ij}(\xi,\omega^s)\frac{\partial^2\theta^0(x)}{\partial x_i\partial x_j}+k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\frac{W_{\alpha_1}(x)}{\partial x_i} \\
& +\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\right)\frac{\partial W_{\alpha_1}(x)}{\partial x_j} \\
& +\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\right)W_{\alpha_1\alpha_2}(x)=f(x)
\end{aligned}
\tag{12}
$$

Although the random distribution in Q^s and Q^t $(s\neq t)$ is subjected to the same probability space, they are independent of each other. For a specific realization $\omega^s\in P$, and then the random functions such as $k_{ij}(\xi,\omega^s)$, $N^s_{\alpha_1}(\xi,\omega^s)$, $N^s_{\alpha_1\alpha_2}(\xi,\omega^s)$ are not periodic. Therefore it is necessary to construct one new approach different for periodic cases to obtain the homogenized heat transfer coefficients and the homogenized heat transfer equations of (2).

Suppose that the homogenized coefficients $\hat{k}_{ij},(i=1,2,\cdots,n)$ is constant in global Ω. Then (12) can be rewritten as follows.

$$
\begin{aligned}
& k_{ij}(\xi,\omega^s)\frac{\partial^2\theta^0(x)}{\partial x_i\partial x_j}+k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\frac{W_{\alpha_1}(x)}{\partial x_i} \\
& +\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\right)\frac{\partial W_{\alpha_1}(x)}{\partial x_j} \\
& +\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\right)W_{\alpha_1\alpha_2}(x) \\
& -\hat{k}_{ij}\frac{\partial^2\theta^0(x)}{\partial x_i\partial x_j}+\hat{k}_{ij}\frac{\partial^2\theta^0(x)}{\partial x_i\partial x_j}=f(x)
\end{aligned}
\tag{13}
$$

Now let $W_{\alpha_1\alpha_2}(x)=\dfrac{\partial^2\theta^0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}$, and rewrite subscripts in (13), owing to the symmetry of the matrix $(k_{ij}(\xi,\omega^s))_{(n\times n)}$, we obtain

$$
\begin{aligned}
& \frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\right)-\hat{k}_{\alpha_2\alpha_1}+k_{\alpha_2\alpha_1}(\xi,\omega^s) \\
& +k_{\alpha_2 j}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}+\frac{\partial}{\partial \xi_i}\left(k_{i\alpha_2}(\xi,\omega)N^s_{\alpha_1}(\xi,\omega^s)\right)\frac{\partial^2\theta^0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \\
& +\hat{k}_{ij}\frac{\partial^2\theta^0(x)}{\partial x_i\partial x_j}=f(x)
\end{aligned}
\tag{14}
$$

Further, suppose that $\theta^0(x)$ is the solution of the following homogenized problems

$$
\begin{cases}
\hat{k}_{ij}\frac{\partial^2\theta^0(\mathbf{x})}{\partial x_i\partial x_j}=f(x) & x\in\Omega, \\
\theta^0(x)=\bar{\theta}(x) & x\in\partial\Omega
\end{cases}
\tag{15}
$$

where $\theta^0(x)$ is called as the global homogenized solution on Ω. Since $\frac{\partial^2\theta^0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}$ $(\alpha_1,\alpha_2=1,\ldots,n)$ are not always identical with zero, then the following equations can be obtained

$$\frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\right) = \hat{k}_{\alpha_2\alpha_1} - k_{\alpha_2\alpha_1}(\xi,\omega^s)$$
$$- k_{\alpha_2 j}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j} - \frac{\partial}{\partial \xi_i}\left(k_{i\alpha_2}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\right) \tag{16}$$

In order to uniquely define $N^s_{\alpha_1\alpha_2}(\xi,\omega^s)$ in Q^s, we supposed that $N^s_{\alpha_1\alpha_2}(\xi,\omega^s)$ is independent of $N^t_{\alpha_1\alpha_2}(\xi,\omega^t)$ $(s \neq t)$ and set

$$N^s_{\alpha_1\alpha_2}(\xi,\omega^s) = 0 \qquad \xi \in \partial Q^s$$

Thus, for any $\omega^s \in$P , in each Q^s one obtain the following problems

$$\begin{cases} \frac{\partial}{\partial \xi_i}\left(k_{ij}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1\alpha_2}(\xi,\omega^s)}{\partial \xi_j}\right) = \hat{k}_{\alpha_2\alpha_1} - k_{\alpha_2\alpha_1}(\xi,\omega^s) \\ \qquad - k_{\alpha_2 j}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j} - \frac{\partial}{\partial \xi_j}\left(k_{i\alpha_2}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\right) \\ N^s_{\alpha_1\alpha_2}(\xi,\omega^s) = 0 \qquad \xi \in \partial Q^s \end{cases} \tag{17}$$

The equations (17) represents the basic properties of composite materials in the domain εQ^s. Therefore, the integration of the right terms of this equations in Q^s will be zero. It can lead to the following equations :

$$\int_P\left(\int_{Q^s}\hat{k}_{\alpha_2\alpha_1}\mathbf{d}\xi - \int_{Q^s}k_{\alpha_2\alpha_1}(\xi,\omega^s)\mathbf{d}\xi - \int_{Q^s}k_{\alpha_2 j}(\xi,\omega^s)\frac{\partial N^s_{\alpha_1}(\xi,\omega^s)}{\partial \xi_j}\mathbf{d}\xi \right.$$
$$\left. - \int_{Q^s}\frac{\partial}{\partial \xi_i}\left(k_{i\alpha_2}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\right)\mathbf{d}\xi\right)\mathbf{d}\mu(\omega^s) = 0 \tag{18}$$

Considering the fixed boundary condition of equations(11), it is obvious to see

$$\int_{Q^s}\frac{\partial}{\partial \xi_i}\left(k_{i\alpha_2}(\xi,\omega^s)N^s_{\alpha_1}(\xi,\omega^s)\right)\mathbf{d}\xi = 0$$

Let

$$\hat{k}_{ij}(\omega^s) = \frac{1}{|Q^s|}\int_{Q^s}\left(k_{ij}(\xi,\omega^s) + k_{ip}(\xi,\omega^s)\frac{\partial N^s_j(\xi,\omega^s)}{\partial \xi_p}\right)\mathbf{d}\xi \tag{19}$$

If the expectation value of $\hat{k}_{ij}(\omega)$ uniquely exists, based on Kolmogoov' classical strong law of large number we can obtain the expected homogenized coefficients.

$$\hat{k}_{ij} = \lim_{M\to\infty}\frac{\sum_{s=1}^{M}\hat{k}_{ij}(\omega^s)}{M} \tag{20}$$

As for how to prove that the expectation value of $\hat{k}_{ij}(\omega)$ $(\omega \in P)$ uniquely exists, seeing the following theorem.

Theorem 4.1: If $\hat{k}_{ij}(\omega)$ is defined as above, the expectation value of $\hat{k}_{ij}(\omega)$ exists.

Proof : in lemma3.2 $|k_{ij}(\xi,\omega)| < M_1$ for any $\omega \in P$. Then from (11), this equation has one unique solution $N_{\alpha_1}(\xi,\omega) \in H^1(Q)$ such that

$$||N_{\alpha_1}(\xi,\omega)||_{H^1(Q)} < C|k_{ij}(\xi,\omega)|_{L^\infty(Q)} < CM_1 \tag{21}$$

for any $\omega \in$ P . where C and M_1 are constants and independent of ξ and ω.
From (19) and (21), one obtains

$$\begin{aligned}|\hat{k}_{ij}(\omega)| &= |\frac{1}{|Q|}\int_Q \left(k_{ij}(\xi,\omega) + k_{ip}(\xi,\omega)\frac{\partial N_j(\xi,\omega}{\partial \xi_p})\right)\mathbf{d}\xi| \\ &< \frac{1}{|Q|}(\|k_{ij}(\xi,\omega)\|_{L^\infty(Q)} + \|k_{ip}(\xi,\omega)\|_{L^\infty(Q)}\|N_j(\xi,\omega)\|_{H^1(Q)}) \\ &\leq \frac{1}{|Q|}|M_1 + M_1 * CM_1||Q| = (C+1)M_2 = M_3\end{aligned} \tag{22}$$

where $M_3 = (C+1)M^2$ and M_3 is independent of the ξ, ω. Therefore, from Theorem 3.1 there exists one unique expectation value in the probability space P.

From the process of obtaining the homogenized heat transfer coefficients $\hat{k}_{ij}$, it is easy to prove that this matrix $(\hat{k}_{ij})_{n\times n}$ is symmetry and positive definite. Then homogenized problem (15) is proper, it has an unique solution $\theta^0(x)$ from lax-Mligrim theorem and Poncare's inequalities, and it is obvious to see $\theta^0(x,\omega,P) \in H^2(Q)$.
Summing up ,one obtains the following theorem
Theorem 4.2. The problem 2 of composite materials with the probability distribution of ε-periodicity ($\varepsilon << 1$) has the formal solution as follows:

$$\theta^\varepsilon(x,\omega,P) = \theta^0(x) + \varepsilon N^s_{\alpha_1}(\xi,\omega^s)\frac{\partial \theta^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 N^s_{\alpha_1\alpha_2}(\xi,\omega^s)\frac{\partial^2\theta^0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + \varepsilon^3 P_1(x,\xi,\omega) \tag{23}$$

Where $\theta^0(x)$ is the solution of the problem(15), called as homogenized solution, $N^s_{\alpha_1}(\xi,\omega^s)$, and $N^s_{\alpha_1\alpha_2}(\xi,\omega^s)$ $(\alpha_1,\alpha_2 = 1,\ldots,n)$ are the solutions of the problem (11), (17), respectively.

5 The procedure of MSA computation based on TSA

As for the multi-phase composite materials with random grains/cavities distribution stated previously, the expected homogenized heat transfer coefficients can be evaluated by the multi-scale method based on TSA. It should be pointed out that for the large size range of grains/cavities it is very complicated to numerically generate the distribution inside a typical cell for each sample. In order to reduce the complexity of computation, according to the size grains/cavities they will be divided into m scales with the different probability distributions. Then step by step by generating the random distribution and using TSA to evaluate the expected homogenized heat transfer coefficients in the algorithm of recurrence.

Now let us state the procedure of multi-scale computation of the expected homogenized heat transfer coefficients as follows: First suppose that material coefficients of matrix to be $\{k_{ij}\}$, material coefficients of grains to be $\{k'_{ij}\}$

1. Calculate the expected homogenized heat transfer coefficients $\hat{k}_{ij}(\varepsilon^m)$.

   (a) Based on the statistics character of the m-th scale grains/cavities for $\forall\ \omega^s \in P$, a distribution model of grains/cavities is generated. And then heat transfer coefficients $k^s_{ij}(\frac{x}{\varepsilon^m},\omega^s)$

on $\forall Q^s(\varepsilon^m)$ can be defined

$$k_{ij}(\frac{x}{\varepsilon^m},\omega^s)=\begin{cases} k_{ij}, & x\in\hat{Q}^s(\varepsilon^m)\\ k'_{ij}, & x\in\tilde{Q}^s(\varepsilon^m)\end{cases} \tag{24}$$

where $\hat{Q}^s(\varepsilon^m)\bigcup\tilde{Q}^s(\varepsilon^m)=Q^s(\varepsilon^m)$ and $\hat{Q}^s(\varepsilon^m)\bigcap\tilde{Q}^s(\varepsilon^m)=\phi$,, $\hat{Q}^s(\varepsilon^m)$ is the domain of the matrix in $Q^s(\varepsilon^m)$ and $\tilde{Q}^s(\varepsilon^m)$ is the domain of of grains/cavities in $Q^s(\varepsilon^m)$.

(b) From the heat transfer coefficients $k_{ij}(\frac{x}{\varepsilon^m},\omega^s)$ $(m=1,2,\ldots,r)$ $N^s_{\alpha_1 m}(\xi^m,\omega^s)$ can be obtained on $Q^s(\varepsilon^m)$ by solving the FE equation of problem (15). So the homogenized coefficients $\hat{k}_{ij}(\frac{x}{\varepsilon^m},\omega^s)$ can be calculated from formula (19).

(c) For $\omega^s\in P$, $s=1,2,\ldots,M^m$, (a) and (b) are repeated M^m times. So M^m the homogenized coefficients $\widehat{k}_{ij}(\frac{x}{\varepsilon^m},\omega^s)(s=1,2,\cdots,M^m)$ are obtained. Thus the expected homogenized coefficients $\hat{k}_{ij}(\varepsilon^m)$ can be evaluated.

$$\hat{k}_{ij}(\varepsilon^m)=\frac{\sum_{s=1}^{M^m}\hat{k}_{ij}(\frac{x}{\varepsilon^m},\omega^s)}{M^m} \tag{25}$$

2. For $r=m-1,m-2,\cdots,1$, calculate the expected homogenized coefficients $\hat{k}_{ij}(\varepsilon^r)$ successively and recursively:

(a) Based on the statistics character of the r-th scale grains/cavities, the distribution model of grains/cavities is generated for $\omega^s\in P$. Then $k_{ij}(\frac{x}{\varepsilon^r},\omega^s)$ of heat transfer coefficients inside $Q^s(\varepsilon^r)$ can be defined as follows:

$$k_{ij}(\frac{x}{\varepsilon^r},\omega^s)=\begin{cases} \hat{k}_{ij}(\varepsilon^{r+1}), & x\in\hat{Q}^s(\varepsilon^r)\\ k'_{ij}, & x\in\tilde{Q}^s(\varepsilon^r)\end{cases} \tag{26}$$

where $\hat{Q}^s(\varepsilon^r)\bigcup\tilde{Q}^s(\varepsilon^r)=Q^s(\varepsilon^r)$ and $\hat{Q}^s(\varepsilon^r)\bigcap\tilde{Q}^s(\varepsilon^r)=\phi$, $Q^s(\varepsilon^r)$ denotes the basic configuration , $\hat{Q}^s(\varepsilon^r)$ the domain of matrix and $\tilde{Q}^s(\varepsilon^r)$ the domain of grains/cavities in $Q^s(\varepsilon^r)$.

(b) From the heat transfer coefficients $k_{ij}(\frac{x}{\varepsilon^r},\omega^s)$ the homogenized coefficients in the r-th scale $\hat{k}_{ij}(\frac{x}{\varepsilon^r},\omega^s)$ can be calculated as (b) in step 1.

(c) For $\omega^s\in P$, $s=1,2,\ldots,M^r$,the (a), (b) are repeated M^r times. So M^r homogenized coefficients $\hat{k}_{ij}(\frac{x}{\varepsilon^r},\omega^s)(s=1,2,\cdots,M^r)$ are obtained. Then the expected homogenized coefficients $\widehat{k}_{ij}(\varepsilon^r)$ can be evaluated

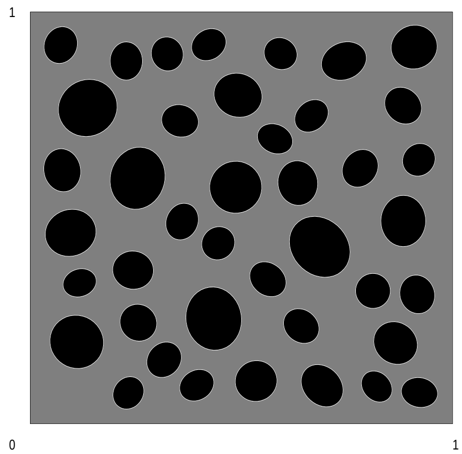
$$\hat{k}_{ij}(\varepsilon^r)=\frac{\sum_{s=1}^{M^r}\hat{k}_{ij}(\frac{x}{\varepsilon^r},\omega^s)}{M^r} \tag{27}$$

3. At last, the homogenized coefficients $\hat{k}_{ij}(\varepsilon^1)$ are the expected homogenized heat transfer coefficients of the composite materials in Ω.

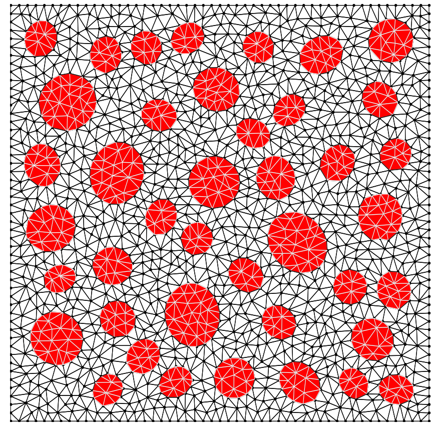
6 Numerical Experiment

To verify the previous algorithm, the numerical results of the computation of homogenized coefficients for one distribution model of random reinforce materials in 2 scales in 2-D case are given in this section.

In the example, we simulate the concrete materials made up of cement, sands,and rock grains. In this composite materials, rock grains have thirty percents. This grains are supposed to be divided into two scales Figure a, Figure b, according to the long axis of the grains shown in Figure 1. The length of the statistic screen is ε^1, and ε^2,respectively. The long axis a ,the short axis b and the inclination θ



G1 stones In ε^3-Screen



G2 the triangle mesh In ε^3-Screen

Table 1 The distributions of the sands and the rock grains in the screens

The sands		The rock grains	
θ	$[0, 2\pi]$	θ	$[0, 2\pi]$
a	$[0.03, 0.08]$	a	$[0.3, 0.8]$
b	$[0.02, 0.06]$	b	$[0.2, 0.6]$

Table 2 The heat transfer coefficients of cement−sands and rock grains

The concrete materials of sands	The Rock grains
$\begin{pmatrix} 2.24 & 0 \\ 0 & 2.24 \end{pmatrix}$	$\begin{pmatrix} 39.00 & 0 \\ 0 & 39.00 \end{pmatrix}$

Table 3 The expected homogenized results of each scale for ED

$$\begin{pmatrix} 23.34 & 0 \\ 0 & 24.83 \end{pmatrix}$$

of the long axis are subjected to average distribution in a certain interval, shown in Table 1, and the heat transfer coefficients of the sand and the rock grains is shown in the Table 2.

By the above specified data and the simulation method of composite material, the grains in each screen for one sample can be easily generated. The screen with 43 grains is generated as shown in Figure G_1 and the domain is partitioned into FE set shown in Figure G_2. We generate 50 samples. Then the expected homogenized coefficients have been calculated by the procedure given in section 5. The final result is given in Table 3. It shows that the composite materials of grain with uniform distribution characteristic has isotropic heat transfer performance.

7 Conclusion

In this paper, we study the heat transfer equations of the composite materials with random grains/cavities and give one MSA method to compute the expected heat transfer coefficients based on TSA. From the numerical results, the validity of this method is obvious. The method can also be extended to deal with other physical fields problems.

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