

Multi-scale Computational Method for Mechanics Parameters of Composite Materials With Random Grain Distribution

Li Youyun Cui Junzhi

The Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100080, China

Tel. : (+8610) 6256 8715 (o), Fax. : (+8610) 6254 2285,

E-mail: lyy@lsec.cc.ac.cn, cjz@lsec.cc.ac.cn

Abstract In this paper, multi-scale method(MSA) for the performance computation of composite materials with random grain distribution is presented. First the representation of the materials with random grain distribution is briefly described. Then the two-scale analysis(TSA) expression of mechanics behavior for the structure of composite materials with random distribution and the procedure of MSA computation based on TSA is carefully discussed. Finally the numerical result for mechanical parameters of the composite materials with random grain distribution is shown. The numerical result shows that MSA is a very effective method for the performance prediction of the material with random distribution.

keywords multi-scale computational method, random grain distribution, composite material, expected homogenized coefficients.

1 Introduction

With the rapid advance of material science, engineering science, especially computing technology, the computational material science has been developing very fast. A variety of numerical methods for the prediction of physical and mechanical performance of materials are developed, refer to [4-8], in last decade. In this paper we are concerned with the computational method of mechanics parameters for composite materials with random grain distribution.

The composite materials can be divided into two classes according to the basic configuration: composite materials with periodic configurations, such as braided composites, and composite materials with random distribution, such as concrete materials. Due to the difference of configurations it is necessary to make use of different numerical methods to evaluate the physical and mechanical performance of materials.

Based on the homogenization method proposed by J.L. Lions, O.A. Oleinik and others scientists, multi-scale analysis (MSA) for the structure of composite materials with periodic configurations was proposed in [5,6,7,8]. Up to now, the papers on MSA method for mechanics parameters of composite materials with random distribution are very few. Therefore, In this paper, MSA method concerning composite material with random distribution and related structure will be presented.

The remainder of this paper is outlined as follows. In section 2 the representation of the materials with random grain distribution are briefly described, and the computer simulation for random grain distribution in the domain is given in section 3. Section 4 is devoted to two-scale analysis (TSA) expression for the mechanics behavior of the structure made from composite materials with random distribution. In section 5 some probability theorems with random configuration are discussed. In section 6 based on TSA the procedure of MSA computation is briefly stated. In section 7 the numerical result for mechanical parameters of three-phase composite materials with random grain distribution is shown.

2 Representation of composite materials with random grain distribution

The configuration of composite materials with random grain distribution, such as concrete and multi-phase composite materials, can be represented as follows:

all of the grains are considered as different scale ellipses in investigated structure, and then all of the ellipses are divided into several classes according to their scales(long axis), in this paper, suppose that the difference between the long axis and the short axis in each ellipse is not very large.

One can consider that grains inside a statistical screen with certain scale for each classification of grains are subjected to the certain probability distribution. In this way we suppose that there exist $\varepsilon^r (r = 1, \dots, m)$ and $\varepsilon^r \gg \varepsilon^{r+1}$, the grains with the scale $l^r (\varepsilon^r > l^r)$ are subjected to certain probability distribution within the statistical screen whose size is ε^r . Therefore the mechanical coefficients of a sample for the composite materials with random grains distribution can be expressed as $\{a_{ijk}(\frac{x}{\varepsilon^r}, \omega), \omega \in P\}$, respectively, where P denotes the probability space.

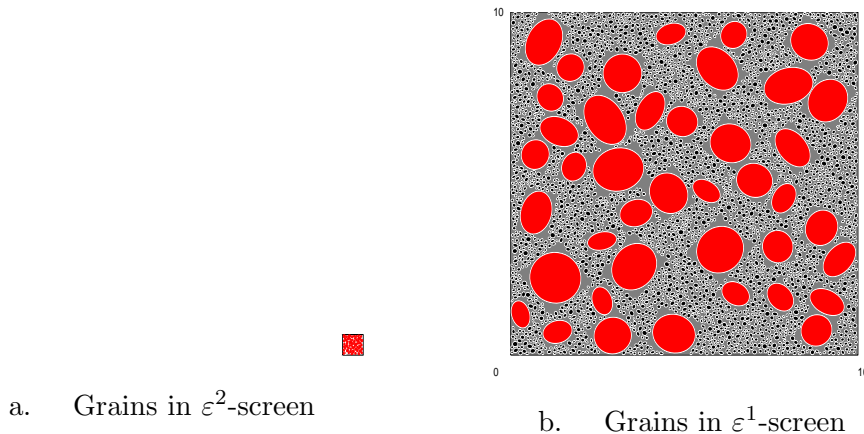


Figure 1. The grain statistical model of the two screens with different scales for the three-phase composite materials

Now let us consider a structure of multi-phase composite materials, which consists of multi-scale grains. For example, for the concrete material one supposes that there are three scales of grains, with the long axis's length of 0.005m-0.015m, 0.015m-0.04m, 0.04m-0.08m, denoted by l^3 , l^2 , and l^1 in the structure. They are subjected to the certain different probability models in their scales, respectively, shown in Figure 1. We can choose that $\varepsilon^1 = 0.2m$, $\varepsilon^2 = 0.08m$, and $\varepsilon^3 = 0.03m$. Apparently, ε^{r+1} is small scale in comparison with ε^r , and then $\varepsilon^1 > \varepsilon^2 > \varepsilon^3 > l^3 > 0$.

From the engineering survey of composite materials by the fitting method of statistic data, the probability model P of the grains distribution inside the structure can be set up:

1. The large grains, whose long axis $l \geq \alpha L$, and L is the scale of structure Ω , are determined, generally, by choosing $\alpha \approx 10^{-1}$.
2. Choose the screen with the size ε^r ($r = 1, \dots, m$), which satisfy $\alpha L > \varepsilon^r > \dots > \varepsilon^{r+1} > \varepsilon^m > 0$, the statistical model of grains with l^r , satisfying $\varepsilon^r > l^r$, can be determined in the following way:
 - (a) Specify the distribution density of grains or the ratio of total volume of all grains, and the distribution model of central points of them, for example, uniform distribution in the ε^r screen.
 - (b) Specify the distribution model of the long axis, and the middle axis and the short axis, and the distribution model of the directions for the long axis and the middle axis.

For certain size ε^r ($1 \leq r \leq m$), from each sample $\omega \in P$, one sample can be obtained, in which there exist a number of grains. For example, a sample for three-phase composite materials is shown in Figure 1. And then the periodically statistic distribution $\{a_{ijk}(\frac{x}{\varepsilon^r}, \omega)\}$ on physical and mechanical parameters are set up.

3 The computer simulation for the materials with grain random distribution

3.1 The algorithms generating the random distribution of the ellipses

In practical computation, the ratio of the basic components of the materials is specified by the statistic methods at first. Then the computer generates the simulation domain and the distribution. For simplification, only 2D case is considered in this paper. So the domain occupied by simulated materials is a 2D rectangle. It can be generated by the computer as follows:

1. The domain for the simulation belongs to the 2-dimension rectangle $[x_{min}, x_{max}] \times [y_{min}, y_{max}]$.
2. Suppose that the central points of the elliptic grains (x_0, y_0) are subjected to the uniform distribution. and the directions θ of the long axis of the elliptic grains are uniform distribution in a interval $[\theta_d, \theta_e]$, the length of long axis are subjected to the normal distribution $N(\alpha, \delta_\alpha)$, and the short axis b to the normal distribution $N(\beta, \delta_\beta)$.

For three dimension, one also can generate the distribution of elliptic grains inside the certain statistic screen as two dimensions.

The algorithms generating the random number conforming with some special distributions are given as below:

1. Exponent distribution. suppose probability density function is:

$$f(x) = \lambda e^{-\lambda x}$$

The random number conforming to the exponent distribution can be gained : $\xi_i = -\frac{\ln r_i}{\lambda}$, r_i conforming to the uniform distribution in $[0, 1]$.

2. Normal distribution. Suppose the random number ξ subjected to $N(\mu_x, \theta_x)$, then

$$\xi = \left(\sum_{i=1}^{i=12} r_i - 6.0 \right) \sigma_x + \mu_x$$

In the above equation, σ_x is covariance, μ_x is the expectation, r_i is the random number subjected to uniform distribution in $[0, 1]$.

3. Logarithm normal distribution. suppose the probability density function:

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-0.5 \left(\frac{\ln x - \mu_x}{\sigma_x} \right)^2}$$

the random number :

$$\xi = e^{\mu_y + \theta_y (\sum_{i=1}^{i=12} r_i - 6.0)}$$

where $\theta_y = \sqrt{\ln(1 + (\frac{\sigma_x}{\mu_x})^2)}$, $\mu_y = \ln \mu_x - 0.5 \sigma_y^2$, r_i is the random number subjected to uniform distribution $[0, 1]$.

For other special distribution, please see the paper [2].

Now let us turn to describe the algorithm to generate the ellipse distribution for a sample. In order to obtain the random distribution ellipses within a screen, nn denotes the number

of the ellipses, and k denotes the number that the ellipses are rejected to locate themselves into the screen while they are generated, because there are some ellipses formerly generated occupying the positions. K is the maximum number of that the ellipse is rejected into the screen. If $nn > N$, which is the total number of admissible ellipses in the screen or if $k > K$, the algorithm will end.

1. Input the distribution model of ellipses and generate the random number conforming to the distribution.
 - (a) Input the distribution parameters of ellipses, generate the total number N of admissible ellipses and the maximum number K of rejected ellipses, let $nn=1$; $k=0$.
 - (b) Generate the random number based on the above distribution.
2. Generate the ellipses in the screen randomly.
 - (a) If $nn > N$ or $k > K$, go to 5, otherwise, go to (b).
 - (b) Generate the central point (x_0, y_0) , the long axis a , the short axis b , the direction of the long axis θ according to their distributions.
 - (c) If $nn=1$, go to step 4, otherwise go to step 3.
3. Differentiate the ellipse to satisfy the conditions.
 - (a) if the new ellipse and the old ellipses have the intersection, $k=k+1$, go to step 2.
 - (b) if the old ellipses in the screen include the new ellipse or the new include the old, $k=k+1$, go to step 2. Otherwise go to step 4.
4. $nn=nn+1$; if $nn < N$, go to step 2. Otherwise go to step 5
5. Output the ellipse distribution in the screen.

3.2 The definition of the probability space

Each ellipse in two dimensions is established by five parameters: the long axis a , the short axis b , the direction θ of the long axis, the center (x_0, y_0) of the ellipses. Their probability density functions are defined as $f_a(x)$, $f_b(x)$, $f_\theta(x)$, $f_{x_0}(x)$, $f_{y_0}(x)$, respectively.

Based on the above supposition, we can obtain a samples ω^s , which is one vector,

$$\omega^s = (a_1^s, b_1^s, \theta_1^s, x_1^s, y_1^s, \dots, a_N^s, b_N^s, \theta_N^s, x_N^s, y_N^s) \in P$$

where P denotes the probability space, N denotes the maximum numbers of grains in a screen. Therefore, for each sample the material parameters $a_{ijhk}(x, y, \omega^s) (i, j, h, k = 1, 2, \dots, n)$ can be defined:

$$a_{ijhk}(x, y, \omega^s) = \begin{cases} a_{ijhk}^1 & \text{if } p(x, y) \in e_{i_1} \quad (i_1 = 1, \dots, N) \\ a_{ijhk}^2 & \text{if } p(x, y) \in Q^s - \cup_{i_1=1}^N e_{i_1} \end{cases}$$

where e_{i_1} is the i_1 -th ellipse in the Q^s and is defined:

$$\frac{(x \cos \theta_{i_1}^s + y \sin \theta_{i_1}^s - x_{i_1}^s \cos \theta_{i_1}^s - y_{i_1}^s \sin \theta_{i_1}^s)^2}{(a_{i_1}^s)^2} + \frac{(y \cos \theta_{i_1}^s - x \sin \theta_{i_1}^s + x_{i_1}^s \sin \theta_{i_1}^s - y_{i_1}^s \cos \theta_{i_1}^s)^2}{(b_{i_1}^s)^2} \leq 1$$

where a_{ijhk}^1, a_{ijhk}^2 are the constants, and $\| \max\{a_{ijhk}^1, a_{ijhk}^2\} \| < M_2, M_2 > 0$. It is obvious to see that $a_{ijhk}(x, y, \omega) (i, j, h, k = 1, 2, \dots, n)$ are the random functions for any $\omega \in P$.

4 The two-scale formulation in probability space

In this section, it is supposed that the structure Ω is only composed of entire ε -cells.

$$\Omega = \bigcup_{s \in T} \varepsilon Q^s$$

where εQ^s denotes a statistic screen and Q^s is a 1-square. Let $\{a_{ijhk}^\varepsilon(x, \omega)\}$ be the tensor of material parameters as the above, where $\omega \in P$ is a random variable to show the character of grains in each Q^s .

For the given structure Ω and the tensor of material parameters $a_{ijhk}^\varepsilon(\xi, \omega)$, let's consider the following elasticity problem:

$$\begin{cases} \frac{\partial}{\partial x_j} \left[a_{ijhk}^\varepsilon(x, \omega, P) \frac{1}{2} \left(\frac{\partial u_h^\varepsilon(x, \omega, P)}{\partial x_k} + \frac{\partial u_k^\varepsilon(x, \omega, P)}{\partial x_h} \right) \right] = f_i(x) & x \in \Omega \\ \mathbf{u}^\varepsilon(x, P) = \bar{\mathbf{u}} & x \in \partial\Omega \end{cases} \quad (1)$$

In common case, for $\forall \omega \in P$, the tensor of material parameters satisfies the below Lemma.

Lemma 1. Material constants $\{a_{ijhk}^\varepsilon(x, \omega, P)\}$ are bounded, measurable and symmetric i.e

$$a_{ijhk}^\varepsilon(x, \omega, P) = a_{ijkh}^\varepsilon(x, \omega, P) = a_{jihk}^\varepsilon(x, \omega, P) \quad (2)$$

and satisfy the $E(\mu_1, \mu_2)$ condition

$$\mu_1 \eta_{ih} \eta_{ih} \leq a_{ijhk}^\varepsilon(x, \omega, P) \eta_{ih} \eta_{jk} \leq \mu_2 \eta_{ih} \eta_{ih} \quad (3)$$

where $\{\eta_{ih}\}$ is a symmetric matrix, and μ_1, μ_2 are constants with $0 < \mu_1 < \mu_2$.

In this paper it is supposed that $f(x) \in C^\infty(\Omega)$. If $f(x) \in L^2(\Omega)$, one can construct a smooth operator $\delta : f(x) \rightarrow \delta f(x) \in C^\infty(\Omega)$ satisfying $\|\delta f(x) - f(x)\| \rightarrow 0$ as $\delta \rightarrow 0$.

From (2), (3), Korn's inequalities and Lax-Millgram Lemma, for $\forall \omega \in P$ the problem (1) has one unique solution $\mathbf{u}^\varepsilon(x, \omega, P)$. It is well known that the displacement and stresses in the structure Ω both depend on the mechanical behaviors of whole structure and the random distribution in each Q^s at the same time. So the displacement can be denoted as $\mathbf{u}^\varepsilon(x, \omega, P) = \mathbf{u}^\varepsilon(x, \xi, \omega, P)$, where the vector x represents the mechanical behaviors of the whole structure and the global coordinates of the structure, ξ denotes the effects of random configurations in εQ^s and the local coordinates.

In order to obtain the two-scale expression of the displacement, it is assumed that $\mathbf{u}^\varepsilon(x, \omega, P)$ can be expanded into the series of the following form:

$$\begin{aligned} \mathbf{u}^\varepsilon(x, \omega, P) = & \mathbf{u}^0(x) + \varepsilon \mathbf{N}_{\alpha_1}^s(\xi, \omega^s) \mathbf{W}_{\alpha_1}(x) + \varepsilon^2 \mathbf{N}_{\alpha_1 \alpha_2}^s(\xi, \omega^s) \mathbf{W}_{\alpha_1 \alpha_2}(x) \\ & + \varepsilon^3 \mathbf{P}_1(x, \xi, \omega, P) \quad x \in \Omega, \quad \xi \in Q^s \quad s = 1, 2, \dots, M \end{aligned} \quad (4)$$

where $\mathbf{N}_{\alpha_1}^s(\xi, \omega^s)$, and $\mathbf{N}_{\alpha_1 \alpha_2}^s(\xi, \omega^s)$ are matrix-valued functions, and $\mathbf{W}_{\alpha_1}(x)$, $\mathbf{W}_{\alpha_1 \alpha_2}(x)$ are the vector valued functions and they are expressed as follows:

$$\mathbf{N}_{\alpha_1}^s(\xi, \omega^s) = \begin{pmatrix} N_{\alpha_1 11}^s(\xi, \omega^s) & \cdots & N_{\alpha_1 1n}^s(\xi, \omega^s) \\ \vdots & & \vdots \\ N_{\alpha_1 n1}^s(\xi, \omega^s) & \cdots & N_{\alpha_1 nn}^s(\xi, \omega^s) \end{pmatrix}, \mathbf{W}_{\alpha_1}(x) = \begin{pmatrix} W_{\alpha_1 1}(x) \\ \vdots \\ W_{\alpha_1 n}(x) \end{pmatrix} \quad (5)$$

$$\mathbf{N}_{\alpha_1 \alpha_2}^s(\xi, \omega^s) = \begin{pmatrix} N_{\alpha_1 \alpha_2 11}^s(\xi, \omega^s) & \cdots & N_{\alpha_1 \alpha_2 1n}^s(\xi, \omega^s) \\ \vdots & & \vdots \\ N_{\alpha_1 \alpha_2 n1}^s(\xi, \omega^s) & \cdots & N_{\alpha_1 \alpha_2 nn}^s(\xi, \omega^s) \end{pmatrix}, \mathbf{W}_{\alpha_1 \alpha_2}(x) = \begin{pmatrix} W_{\alpha_1 \alpha_2 1}(x) \\ \vdots \\ W_{\alpha_1 \alpha_2 n}(x) \end{pmatrix} \quad (6)$$

From (4), (5), (6) ,

$$\begin{aligned} u_h^\varepsilon(x, \omega, P) = & u_h^0(x) + \varepsilon N_{\alpha_1 h m}^s(\xi, \omega^s) W_{\alpha_1 m}(x) + \varepsilon^2 N_{\alpha_1 \alpha_2 h m}^s(\xi, \omega^s) W_{\alpha_1 \alpha_2 m}(x) \\ & + \varepsilon^3 P_{1h}(x, \xi, \omega, P), h = 1, 2, \dots, n \end{aligned} \quad (7)$$

Where $(h, \alpha_1, \alpha_2, m = 1, 2, \dots, n)$. Due to $\xi = x/\varepsilon$, respecting

$$\frac{\partial u_i^\varepsilon(x, \omega^s)}{\partial x_j} = \frac{\partial u_i(x, \xi, \omega^s)}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial u_i(x, \xi, \omega^s)}{\partial \xi_j} \quad (8)$$

one obtains that:

$$\begin{aligned}
& a_{ijhk}(\xi, \omega, P) \frac{1}{2} \left(\frac{\partial u_h^\varepsilon(x, \omega, P)}{\partial x_k} + \frac{\partial u_k^\varepsilon(x, \omega, P)}{\partial x_h} \right) = a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\
& + a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) W_{\alpha_1 m}(x) \\
& + \frac{1}{2} \varepsilon a_{ijhk}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_k} \\
& + \frac{1}{2} \varepsilon a_{ijhk}(\xi, \omega^s) N_{\alpha_1 km}^s(\xi, \omega^s) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_h} \\
& + \varepsilon a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) W_{\alpha_1 \alpha_2 m}(x) \\
& + \frac{1}{2} \varepsilon^2 a_{ijhk}(\xi, \omega^s) N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s) \frac{\partial W_{\alpha_1 \alpha_2 m}(x)}{\partial x_k} \\
& + \frac{1}{2} \varepsilon^2 a_{ijhk}(\xi, \omega^s) N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s) \frac{\partial W_{\alpha_1 \alpha_2 m}(x)}{\partial x_h} + \varepsilon^3 T(x, \xi, \omega, P)
\end{aligned} \tag{9}$$

where $T(x, \xi, \omega, P) = a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial P_{1h}(x, \xi, \omega, P)}{\partial x_k} + \frac{\partial P_{1k}(x, \xi, \omega, P)}{\partial x_h} \right)$

From (9) and (1), it leads to the following equations:

$$\begin{aligned}
& \frac{\partial}{\partial x_j} \left[a_{ijhk}(\xi, \omega, P) \frac{1}{2} \left(\frac{\partial u_h^\varepsilon(x, \omega, P)}{\partial x_k} + \frac{\partial u_k^\varepsilon(x, \omega, P)}{\partial x_h} \right) \right] = \varepsilon^{-1} \frac{\partial a_{ijhk}(\xi, \omega^s)}{\partial \xi_j} \frac{1}{2} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\
& + a_{ijhk}(\xi, \omega^s) \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\
& + \varepsilon^{-1} \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] W_{\alpha_1 m}(x) \\
& + a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \frac{W_{\alpha_1 m}(x)}{\partial x_j} \\
& + \frac{1}{2} \frac{\partial}{\partial \xi_j} (a_{ijhk}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s)) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_k} + \frac{1}{2} \frac{\partial}{\partial \xi_j} (a_{ijhk}(\xi, \omega^s) N_{\alpha_1 km}^s(\xi, \omega^s)) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_h} \\
& + \frac{1}{2} \varepsilon a_{ijhk}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \frac{\partial^2 W_{\alpha_1 m}(x)}{\partial x_k \partial x_j} + \frac{1}{2} \varepsilon a_{ijhk}(\xi, \omega^s) N_{\alpha_1 km}^s(\xi, \omega^s) \frac{\partial^2 W_{\alpha_1 m}(x)}{\partial x_h \partial x_j} \\
& + \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] W_{\alpha_1 \alpha_2 m}(x) \\
& + \varepsilon a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \frac{W_{\alpha_1 \alpha_2 m}(x)}{\partial x_j} \\
& + \varepsilon \frac{1}{2} \frac{\partial}{\partial \xi_j} \left(a_{ijhk}(\xi, \omega^s) N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s) \right) \frac{\partial W_{\alpha_1 \alpha_2 m}(x)}{\partial x_k}
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon \frac{1}{2} \frac{\partial}{\partial \xi_j} \left(a_{ijhk}(\xi, \omega^s) N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s) \right) \frac{\partial W_{\alpha_1 \alpha_2 m}(x)}{\partial x_h} \\
& + \varepsilon^2 \frac{1}{2} a_{ijhk}(\xi, \omega^s) N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s) \frac{\partial^2 W_{\alpha_1 \alpha_2 m}(x)}{\partial x_k \partial x_j} \\
& + \varepsilon^2 \frac{1}{2} a_{ijhk}(\xi, \omega^s) N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s) \frac{\partial^2 W_{\alpha_1 \alpha_2 m}(x)}{\partial x_h \partial x_j} \\
& + \varepsilon^3 V(x, \xi, \omega, P) = f_i(x)
\end{aligned} \tag{10}$$

where $\varepsilon^3 V(x, \xi, \omega, P) = \varepsilon^3 \frac{\partial T(x, \xi, \omega, P)}{\partial x_j}$

Since the equation (10) holds for any $\varepsilon > 0$, comparing the coefficient of the ε^{-1} in the equation (10), we obtain the following equations.

$$\begin{aligned}
& \frac{\partial a_{ijhk}(\xi, \omega^s)}{\partial \xi_j} \frac{1}{2} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\
& + \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] W_{\alpha_1 m}(x) = 0
\end{aligned} \tag{11}$$

setting $W_{\alpha_1 m}(x) = \frac{\partial u_m^0(x)}{\partial x_{\alpha_1}}$, and making use of the symmetry of $\{a_{ijhk}^\varepsilon\}$, by virtue of (11), one obtains that:

$$\begin{aligned}
& \left(\frac{1}{2} \frac{\partial a_{ijm\alpha_1}(\xi, \omega^s)}{\partial \xi_j} + \frac{1}{2} \frac{\partial a_{ij\alpha_1 m}(\xi, \omega^s)}{\partial \xi_j} \right) \frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} \\
& + \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] \frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} = 0
\end{aligned} \tag{12}$$

It is noted that $\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}}$ does not always equal to zero onto Ω , and the material parameters in each screen are independent each other. Then the following equations are true.

$$\begin{aligned}
& \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] = - \frac{\partial a_{ij\alpha_1 m}(\xi, \omega^s)}{\partial \xi_j} \\
& \xi \in Q^s, i = 1, \dots, n.
\end{aligned} \tag{13}$$

where $m, \alpha_1=1, \dots, n$. Set $\mathbf{N}_{\alpha_1 m}^s(\xi, \omega^s) = \left(N_{\alpha_1 1m}^s(\xi, \omega^s), \dots, N_{\alpha_1 nm}^s(\xi, \omega^s) \right)^T$.

In order to uniquely define $\mathbf{N}_{\alpha_1 m}^s(\xi, \omega^s)$ onto Q^s , it is necessary to prescribe the boundary conditions. Here, choose such a following boundary condition on ∂Q^s in order to satisfy the boundary condition in (1) for $\mathbf{u}^\varepsilon(\mathbf{x}, \omega, \mathbf{P})$ from the deduction

$$\mathbf{N}_{\alpha_1 m}^s(\xi, \omega^s) = 0, \quad \xi \in \partial Q^s \tag{14}$$

Thus, for any $\omega^s \in P$, in each Q^s one obtains the following problems:

$$\begin{cases} \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] = - \frac{\partial a_{\alpha_1 jm}(\xi, \omega^s)}{\partial \xi_j} & \xi \in Q^s \\ \mathbf{N}_{\alpha_1 m}^s(\xi, \omega^s) = 0 & \xi \in \partial Q^s \end{cases} \quad (15)$$

Based on Lax-Millgram Lemma and Korn's inequality, by virtue to lemma 1, it is easy to see that above equation has the unique solution.

Since the right of the equation (10) is independent of ε , we can set that the coefficients of ε^0 in the left of the equation (10) equal to $f_i(x)$, i.e

$$\begin{aligned} & a_{ijhk}(\xi, \omega^s) \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\ & + a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \frac{W_{\alpha_1 m}(x)}{\partial x_j} \\ & + \frac{1}{2} \frac{\partial}{\partial \xi_j} \left(a_{ijhk}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_k} \\ & + \frac{1}{2} \frac{\partial}{\partial \xi_j} \left(a_{ijhk}(\xi, \omega^s) N_{\alpha_1 km}^s(\xi, \omega^s) \right) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_h} \\ & + \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] W_{\alpha_1 \alpha_2 m}(x) \\ & = f_i(x), \quad i, j, h, k = 1, 2, \dots, n. \end{aligned} \quad (16)$$

Although the grain distribution in different Q^s and Q^t ($s \neq t$) are subjected to the same probability distribution, they are independent of each other. For a specific realization $\omega^s \in P$, the distribution function, such as $a_{ijhk}(\xi, \omega^s)$, $\mathbf{N}_{\alpha_1 m}^s(\xi, \omega^s)$, $\mathbf{N}_{\alpha_1 \alpha_2 m}^s(\xi, \omega^s)$ are not periodic. Therefore it is necessary to construct one new approach different from periodic cases to obtain the homogenized coefficients and the equations.

Suppose that there are the homogeneous coefficients $\hat{a}_{ijhk}(i, j, k, h = 1, \dots, n)$ on global Ω and $\{\hat{a}_{ijhk}\}$ satisfies the requirement of lemma 1. Then (16) can be rewritten into

$$\begin{aligned} & a_{ijhk}(\xi, \omega^s) \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\ & + a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \frac{W_{\alpha_1 m}(x)}{\partial x_j} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \frac{\partial}{\partial \xi_j} (a_{ijhk}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s)) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_k} \\
& + \frac{1}{2} \frac{\partial}{\partial \xi_j} (a_{ijhk}(\xi, \omega^s) N_{\alpha_1 km}^s(\xi, \omega^s)) \frac{\partial W_{\alpha_1 m}(x)}{\partial x_h} \\
& + \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] W_{\alpha_1 \alpha_2 m}(x) \quad (17) \\
& - \hat{a}_{ijhk} \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) + \hat{a}_{ijhk} \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) \\
& = f_i(x)
\end{aligned}$$

Now let $W_{\alpha_1 \alpha_2 m}^s(x) = \frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}}$, and rewrite subscripts in (17), owing to the symmetry of the matrix $(a_{ijkh}(\xi, \omega^s))_{n \times n}$, we obtain

$$\begin{aligned}
& \left\{ \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] - \hat{a}_{i\alpha_2 m \alpha_1} + a_{i\alpha_2 m \alpha_1}(\xi, \omega^s) \right. \\
& \left. + a_{i\alpha_2 hk}(\xi, \omega^s) \frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial}{\partial \xi_j} \left(a_{ijh\alpha_2}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \right\} \frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} \\
& + \hat{a}_{ijhk} \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) = f_i(x) \quad (18)
\end{aligned}$$

Further suppose that the vector valued function $\mathbf{u}^0(x)$ is the solution of following problems

$$\begin{cases} \hat{a}_{ijhk} \frac{1}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial u_h^0(x)}{\partial x_k} + \frac{\partial u_k^0(x)}{\partial x_h} \right) = f_i(x) & x \in \Omega, \\ \mathbf{u}^0(x) = \bar{\mathbf{u}} & x \in \partial\Omega \end{cases} \quad (19)$$

$\mathbf{u}^0(x)$ is called as the global homogenization solution on Ω , then the following equations can be obtained:

$$\begin{aligned}
& \left\{ \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] - \hat{a}_{i\alpha_2 m \alpha_1} + a_{i\alpha_2 m \alpha_1}(\xi, \omega^s) \right. \\
& \left. + a_{i\alpha_2 hk}(\xi, \omega^s) \frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial}{\partial \xi_j} \left(a_{ijh\alpha_2}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \right\} \frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} = 0 \quad (20)
\end{aligned}$$

Since $\frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}}$ ($\alpha_1, \alpha_2, m = 1, \dots, n$) are not always identical with zero and the material parameters in different screens are independent of each other, the equations (20) can be

changed into

$$\begin{aligned} \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] &= \hat{a}_{i\alpha_2 m \alpha_1} \\ - a_{i\alpha_2 m \alpha_1}(\xi, \omega^s) - a_{i\alpha_2 hk}(\xi, \omega^s) \frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} - \frac{\partial}{\partial \xi_j} \left(a_{ijh\alpha_2}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \end{aligned} \quad (21)$$

In order to uniquely define $N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)$ in Q^s , it is necessary to prescribe the boundary conditions. Here, as the same as in (14), let

$$N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s) = 0 \quad \xi \in \partial Q^s$$

Thus, for any $\omega^s \in P$, in each Q^s one obtains the following problems

$$\left\{ \begin{array}{l} \frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \alpha_2 km}^s(\xi, \omega^s)}{\partial \xi_h} \right) \right] = \hat{a}_{i\alpha_2 m \alpha_1} - a_{i\alpha_2 m \alpha_1}(\xi, \omega^s) \\ - a_{i\alpha_2 hk}(\xi, \omega^s) \frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} - \frac{\partial}{\partial \xi_j} \left(a_{ijh\alpha_2}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \\ N_{\alpha_1 \alpha_2 hm}^s(\xi, \omega^s) = 0 \quad \xi \in \partial Q^s \end{array} \right. \quad (22)$$

The equations(22) describe the basic properties of materials in each statistical screen ∂Q^s . Therefore, the integration of the right terms of this equations in Q^s ought to equal to zero. It can lead to the following equations :

$$\begin{aligned} \int_P \left[\int_{Q^s} (\hat{a}_{i\alpha_2 m \alpha_1} \mathbf{d}\xi - \int_{Q^s} a_{i\alpha_2 m \alpha_1}(\xi, \omega^s) \mathbf{d}\xi - \int_{Q^s} a_{i\alpha_2 hk}(\xi, \omega^s) \frac{\partial N_{\alpha_1 hm}^s(\xi, \omega^s)}{\partial \xi_k} \mathbf{d}\xi \right. \\ \left. - \int_{Q^s} \frac{\partial}{\partial \xi_j} \left(a_{ijh\alpha_2}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \mathbf{d}\xi \right] d\omega^s = 0 \end{aligned} \quad (23)$$

Considering the fixed boundary condition of equations(15), it is obvious to see

$$\int_{Q^s} \frac{\partial}{\partial \xi_j} \left(a_{ijh\alpha_2}(\xi, \omega^s) N_{\alpha_1 hm}^s(\xi, \omega^s) \right) \mathbf{d}\xi = 0 \quad (24)$$

Let

$$\hat{a}_{ijhk}(\omega^s) = \frac{1}{|Q^s|} \int_{Q^s} \left(a_{ijhk}(\xi, \omega^s) + a_{ijpq}(\xi, \omega^s) \frac{1}{2} \left(\frac{\partial N_{hpq}^s(\xi, \omega)}{\partial \xi_q} + \frac{\partial N_{hqk}^s(\xi, \omega)}{\partial \xi_p} \right) \right) \mathbf{d}\xi \quad (25)$$

If the expectation value of $\hat{a}_{ijhk}(\omega)$ exists, we can evaluate the expected homogenized coefficients by

$$\bar{a}_{ijhk} = \frac{\sum_{s=1}^M \hat{a}_{ijhk}(\omega^s)}{M} \quad (26)$$

So the following remains is obtained.

$$\bar{a}_{ijhk}(\omega^s) = \hat{a}_{ijhk} - \hat{a}_{ijhk}(\omega^s) \quad in \quad Q^s \quad (27)$$

We will prove that the expectation value of $\hat{a}_{ijhk}(\omega)$ ($\omega \in P$) exists in following section 5 in this paper.

From the process of obtaining the homogenized coefficients \hat{a}_{ijhk} , it is easy to prove that the matrix $(\hat{a}_{ijhk})_{n \times n}$ is the symmetry positive definite. Then homogenized problem (19) is proper, it has a unique solution $\mathbf{u}^0(x)$ from lax-Mligrim theorem and Korn's inequalities, and it is obvious to see $\mathbf{u}^0(x) \in H^2(Q)$.

Summing up, one obtains the following theorem.

Theorem 2. The problem (1) of composite materials with the probability distribution of ε -periodicity ($\varepsilon \ll 1$) has the formal approximate solution as follows:

$$\mathbf{u}^\varepsilon(x, \omega, P) = \mathbf{u}^0(x) + \varepsilon \mathbf{N}_{\alpha_1}^s(\xi, \omega^s) \frac{\partial \mathbf{u}^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 \mathbf{N}_{\alpha_1 \alpha_2}^s(\xi, \omega^s) \frac{\partial^2 \mathbf{u}^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} + \varepsilon^3 \mathbf{P}(x, \xi, \omega, P) \quad (28)$$

Where $\mathbf{u}^0(x)$ is the solution of the problem(19) called as the homogenization solution, $\mathbf{N}_{\alpha_1}^s(\xi, \omega^s)$, and $\mathbf{N}_{\alpha_1 \alpha_2}^s(\xi, \omega^s)$ ($\alpha_1, \alpha_2 = 1, \dots, n.$) are the solutions of the following problem, respectively (15), (22).

5 Some probability theorem in the composite materials with random ellipse grains

According to section 3, the long axis a, the short axis b, the direction θ of the long axis and the center $P(x_0, y_0)$ of the ellipse have the following probability density functions $f_a(x)$, $f_b(x)$, $f_\theta(x)$, $f_{x_0}(x)$, $f_{y_0}(x)$, respectively. And the probability space of them are $P_a, P_b, P_\theta, P_{x_0}, P_{y_0}$. Then we can obtain the following theorem.

lemma 3. If $a_{ijhk}(\xi, \omega)$, $N_{\alpha_1 mh}(\xi, \omega)$, and $\hat{a}_{ijhk}(\omega)$ are the same random variable functions as the above, then there exists one constant $M_1 > 0$ such that $|\hat{a}_{ijhk}(\omega)| < M_1$, for any $\omega \in P$.

Proof : Owing to $|a_{ijhk}(\xi, \omega)| < M_2$ for any $\omega \in Q$, from (15), this equation has one unique solution $N_{\alpha_1 hm}(\xi, \omega) \in H^1(Q)$ such that

$$\|N_{\alpha_1 hm}(\xi, \omega)\|_{H^1(Q)} < C |a_{ijhk}(\xi, \omega)|_{L^\infty(Q)} < CM_2 \quad (29)$$

where C and M_2 are constants and independent of ξ and ω .

From (25) and (29), one can obtain

$$\begin{aligned} |\hat{a}_{ijhk}(\omega)| &= \left| \frac{1}{|Q|} \int_{|Q|} \left(a_{ijhk}(\xi, \omega) + a_{ijpq}(\xi, \omega) \frac{1}{2} \left(\frac{\partial N_{hpq}^s(\xi, \omega)}{\partial \xi_q} + \frac{\partial N_{hqk}^s(\xi, \omega)}{\partial \xi_p} \right) \right) \mathbf{d}\xi \right| \\ &< \frac{1}{|Q|} (\|a_{ijhk}(\xi, \omega)\|_{L^\infty(Q)} + \|a_{ijpq}(\xi, \omega)\|_{L^\infty(Q)} \|N_{\alpha_1 hm}(\xi, \omega)\|_{H^1(Q)}) \\ &\leq \frac{1}{|Q|} |M_2 + M_2 * CM_2| |Q| = (C + 1)M_2^2 = M_1 \end{aligned} \quad (30)$$

where $M_1 = (C + 1)M_2^2$ and M_1 is independent of the ξ, ω .

Theorem 4. If ω is a random variable and $\hat{a}_{ijhk}(\omega)$ is defined as the above. Then $E_\omega \hat{a}_{ijhk}(x)$ exists in the probability space $P = (P_a)^N \times (P_b)^N \times (P_\theta)^N \times (P_{x_0})^N \times (P_{y_0})^N$

proof: From the probability density functions $f_a, f_b, f_\theta, f_{x_0}$ and f_{y_0} , the united probability density function $\omega = (a_1, b_1, \theta_1, x_1, y_1, \dots, a_N, b_N, \theta_N, x_N, y_N)$ is

$$f_{a_1, b_1, \theta_1, \dots, x_N, y_N}(x) = f_a^N(x) \cdot f_b^N(x) \cdot f_\theta^N(x) \cdot f_{x_0}^N(x) \cdot f_{y_0}^N(x) \quad (31)$$

From (31) and lemma 3, one can obtain

$$\begin{aligned} E_\omega \hat{a}_{ijhk}(x) &= \int_P \hat{a}_{ijhk}(\omega) \mathbf{d}P = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \hat{a}_{ijhk}(x) f_{a_1, b_1, \theta_1, \dots, x_N, y_N}(x) \mathbf{d}x \\ &< M_1 \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{a_1, b_1, \theta_1, \dots, x_N, y_N}(x) \mathbf{d}x \\ &< M_1 \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_a^N(x) \mathbf{d}x \dots \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{y_0}^N(x) \mathbf{d}x \\ &< M_1 \end{aligned} \quad (32)$$

Therefore there exist one unique expectation tensor values $E_\omega \hat{a}_{ijhk}(x)$ ($i, j, h, k = 1, 2, \dots, n$) in the probability space P.

Corollary 5. If $(\hat{a}_{ijhk}(\omega^s))$ ($i, j, h, k = 1, 2, \dots, n. s = 1, 2, \dots$) have the expectation value $E_\omega \hat{a}_{ijhk}(x)$ in the probability space and x is the random variable, one obtain

$$\frac{\sum_{s=1}^{n_1} \hat{a}_{ijhk}(\omega^s)}{n_1} \xrightarrow{a.e} E_\omega \hat{a}_{ijhk}(x) \quad (n_1 \rightarrow \infty) \quad (33)$$

Proof: Set $\{\hat{a}_{ijhk}(\omega^s), s \geq 1\}$ are the independent and identical distribution random variable and

$$S_{n_1} = \sum_{s=1}^{n_1} \hat{a}_{ijhk}(\omega^s) \quad (n_1 = 1, 2, \dots).$$

Owing to theorem 3, $|E_\omega \hat{a}_{ijhk}(\omega^s)| < \infty$, set $a_{n_1} = E_\omega \hat{a}_{ijhk}(\omega^1) = a_1$, From Kolmogorov's classical strong law of large numbers, one obtains:

$$\frac{S_{n_1}}{n_1} - a_1 \xrightarrow{a.e} 0 \quad (n_1 \rightarrow \infty)$$

Therefore we have $\frac{S_{n_1}}{n_1} \xrightarrow{a.e} E_\omega \hat{a}_{ijhk}(\omega^1) \quad (n_1 \rightarrow \infty) \quad i.e$

$$\frac{\sum_{s=1}^{n_1} \hat{a}_{ijhk}(\omega^s)}{n_1} \xrightarrow{a.e} E_\omega \hat{a}_{ijhk}(x) \quad (n_1 \rightarrow \infty)$$

Remark 6. Here, we only give the proof of the domain with random grains in two dimension case. For the domain with random grains in three dimensions, one only changes the 5 parameters into the 9 parameters to denote the ellipse, others are the same as above. Therefore, the same probability theorems of three dimensions case also be obtained.

6 The procedure of MSA computation based on TSA

As for the multi-phase composite materials with random elliptical grain distribution stated previously in section 3, its expected homogenized coefficients can be evaluated by the multi-scale method based on TSA. It should be pointed out that for the large size range of grains it is very complicated to numerically generate the grains distribution inside a typical cell for each sample. In order to reduce the complexity of computation, according to the size range of grains the grains will be divided into m scales with the different probability distributions. Then step by step by generating the grain and using TSA to evaluate the expected homogenization coefficients in the algorithm of recurrence.

Now let us state the procedure of Multi-scale computation of the expected homogenized coefficients for the composite materials with multi-scale random grain distribution as follows: First suppose that material coefficients of matrix to be $\{a_{ijhk}\}$, material coefficients of grains to be $\{a'_{ijhk}\}$

1. Calculate the expected homogenized coefficients $\hat{a}_{ijhk}(\varepsilon^m)$.

- (a) Based on the statistics character of the m -th scale grains for $\forall \omega^s \in P$, a distribution model of grains is generated. and then material coefficients $a_{ijhk}^s(\frac{x}{\varepsilon^m}, \omega^s)$ on $\forall Q^s(\varepsilon^m)$ can be defined

$$a_{ijhk}(\frac{x}{\varepsilon^m}, \omega^s) = \begin{cases} a_{ijhk}, & x \in \hat{Q}^s(\varepsilon^m) \\ a'_{ijhk}, & x \in \tilde{Q}^s(\varepsilon^m) \end{cases} \quad (34)$$

where $\hat{Q}^s(\varepsilon^m) \cup \tilde{Q}^s(\varepsilon^m) = Q^s(\varepsilon^m)$ and $\hat{Q}^s(\varepsilon^m) \cap \tilde{Q}^s(\varepsilon^m) = \phi$, $\hat{Q}^s(\varepsilon^m)$ is the domain of the matrix in $Q^s(\varepsilon^m)$ and $\tilde{Q}^s(\varepsilon^m)$ is the domain of of grains in $Q^s(\varepsilon^m)$.

- (b) From the material coefficients $a_{ijhk}^s(\frac{x}{\varepsilon^m}, \omega^s)$ ($m = 1, 2, \dots, r$) and $\mathbf{N}_{\alpha_1 m}^s(\xi^m, \omega^s)$ can be obtained on $Q^s(\varepsilon^m)$ by solving the variational equation of problem (15) by FE method. So the homogenized coefficients $\hat{a}_{ijhk}(\frac{x}{\varepsilon^m}, \omega^s)$ can be calculated from formula (26).

- (c) For $\omega^s \in P$, $s = 1, 2, \dots, M^m$, (a) and (b) are repeated M^m times. So M^m the homogenized coefficients $\hat{a}_{ijhk}(\frac{x}{\varepsilon^m}, \omega^s)$ ($s = 1, 2, \dots, M^m$) are obtained. Thus

the expected homogenized coefficients $\hat{a}_{ijhk}(\varepsilon^m)$ for the materials with the grains whose size is equal to and less than ε^m , can be evaluated.

$$\hat{a}_{ijhk}(\varepsilon^m) = \frac{\sum_{s=1}^{M^m} \hat{a}_{ijhk}^s(\frac{x}{\varepsilon^m}, \omega^s)}{M^m} \quad (35)$$

2. For $r = m - 1, m - 2, \dots, 1$, calculate the expected homogenized coefficients $\hat{a}_{ijhk}(\varepsilon^r)$ successively and recursively:

(a) Based on the statistics character of the r-th scale grains in the materials, a distribution model of grains is generated for $\omega^s \in P$. From this model the distribution function $a_{ijhk}^s(\frac{x}{\varepsilon^r}, \omega^s)$ of material coefficients inside $Q^s(\varepsilon^r)$ can be defined as follows:

$$a_{ijhk}(\frac{x}{\varepsilon^r}, \omega^s) = \begin{cases} \hat{a}_{ijhk}(\varepsilon^{r+1}), & x \in \hat{Q}^s(\varepsilon^r) \\ a'_{ijhk}, & x \in \tilde{Q}^s(\varepsilon^r) \end{cases} \quad (36)$$

where $\hat{Q}^s(\varepsilon^r) \cup \tilde{Q}^s(\varepsilon^r) = Q^s(\varepsilon^r)$ and $\hat{Q}^s(\varepsilon^r) \cap \tilde{Q}^s(\varepsilon^r) = \phi$, $Q^s(\varepsilon^r)$ denotes the basic configuration, $\hat{Q}^s(\varepsilon^r)$ the domain of matrix and $\tilde{Q}^s(\varepsilon^r)$ the domain of grains in $Q^s(\varepsilon^r)$.

(b) From the material coefficients $a_{ijhk}(\frac{x}{\varepsilon^r}, \omega^s)$ and the homogenized coefficients in the r-th scale $\hat{a}_{ijhk}(\frac{x}{\varepsilon^r}, \omega^s)$ can be calculated as (b) in step 1.

(c) For $\omega^s \in P$, $s = 1, 2, \dots, M^r$, the (a), (b) are repeated M^r times. So M^r homogenized coefficients $\hat{a}_{ijhk}(\frac{x}{\varepsilon^r}, \omega^s)$ ($s = 1, 2, \dots, M^r$) are obtained. Then the expected homogenized coefficients $\hat{a}_{ijhk}(\varepsilon^r)$ for the materials with the grains whose size is equal to and less than ε^r , can be evaluated

$$\hat{a}_{ijhk}(\varepsilon^r) = \frac{\sum_{s=1}^{M^r} \hat{a}_{ijhk}(\frac{x}{\varepsilon^r}, \omega^s)}{M^r} \quad (37)$$

3. At last, the homogenized coefficients $\hat{a}_{ijhk}(\varepsilon^1)$ are the expected homogenized coefficients of the composite materials in Ω .

7 Numerical experiment

To verify the previous multi-scale algorithm, the numerical results of the computation of homogenized coefficients for one random distribution model of random ellipse grains in 2-D case are given in this section.

In the example, we simulate the concrete made up of cement, sands and rock grains. In this composite materials, rock grains have about 30 percents. This grains are supposed to be divided into two scales Figure a, Figure b, according to the long axis of the grains shown in

Figure 1, the length of whose statistic screen is ε^1 , and ε^2 , respectively. The long axis a , the short axis b , and the inclination θ are subjected to uniform distribution in a certain interval, shown in Table 1, and the Yongth module of the sand and the rock grains is shown in the Table 2.

By the above specified data and the simulation method of the composite materials, the material coefficient of grains in each screen for one sample can be easily generated. The screen with 43 grains is generated as shown in Figure 2 and the domain with grains is

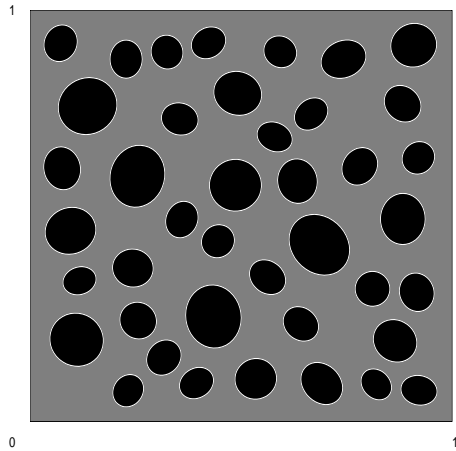


Figure 2 stones in ε^1 -screen

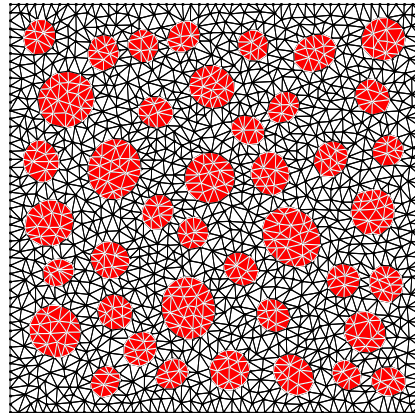


Figure 3 the triangle mesh in ε^1 -screen

Table 1 The **UD** of the sands and the rock grains in the screens

The sands		The rock grains	
θ	$[0, 2\pi]$	θ	$[0, 2\pi]$
a	$[0.03, 0.08]$	a	$[0.3, 0.8]$
b	$[0.02, 0.06]$	b	$[0.2, 0.6]$

Table 2 The material coefficients of cement–sands and rock grains

The concrete materials of sands	The rock grains
$\begin{pmatrix} 8.41593E5 & 7.04189E5 & 0 \\ 7.04189E5 & 8.41593E5 & 0 \\ 0 & 0 & 6.8702E4 \end{pmatrix}$	$\begin{pmatrix} 2.9999E6 & 1.5000E6 & 0 \\ 1.5000E6 & 2.9999E6 & 0 \\ 0 & 0 & 7.5000E5 \end{pmatrix}$

Table 3 The expected homogenized results of each scale for UD

$$\begin{pmatrix} 1.0950E6 & 8.3021E5 & 0 \\ 8.3021E5 & 1.0990E6 & 0 \\ 0 & 0 & 1.2942E5 \end{pmatrix}$$

partitioned into FE set shown in Figure 3. We generate 50 samples at each computation step of previous multi-scale algorithm in the screen. Then the expected homogenized coefficients have been calculated by the procedure given in section 6. The detailed result is given in Table 3. It shows that the composite material of grain reinforcement with uniform distribution characteristic has isotropic mechanics performance.

8 Conclusion

In this paper, we have given a MSA method to compute the mechanics performance of the composite materials with random grain distribution. From the numerical result, the validity of this method is obvious. The method can also be extended to deal with the composite materials with the random short fibre materials, random foams/cavities materials, and physical field problems related to them.

9 Acknowledgements

This work is supported by the Special Funds for Major State Basic Research Project and National Natural Science Foundation of China (19932030).

References

- [1] Li Youyun, Cui Junzhi. The Computer Simulate Method for the domain with large numbers of ellipsoid Grain/Porou Random Distribution and the Improved Algorithm of Auto-Generating Triangulation. *Chinse Journal of Computational mechanics*(2003 to appear)
- [2] Zhangji Xu. Monte Carlo Method. Shanghai technolodge press, 1985.
- [3] A. Paplis. Probability, Random Varible and Stochastic Process. The High Education Press, 1983(in chinese)
- [4] Dierk Raabe. Computational Materials Science. WILEY-VCH verlag Gmbh, 1998
- [5] V.V.Jikov, S.M.Kozlov and O.A. Oleinik. Homogenization of Differential Operators and Integral Functionals. Springer-Verlag, 1994.
- [6] Oleinik, O.A., Shamaev, A.S. and Yosifian. G.A. Mathematical Problems in Elasticity and Homogenization, Amsterdam: North - Holland, 1992

- [7] Cui J.Z, Shih T.M. and Wang Y.L. The Two-Scale Analysis Method for Bodies With Small Periodic Configuration. Invited Paper in CASC97, Sydney, Australia, Proc.of CASC97, *Structural Engineering and Mechanics*, 1997, Vol.7, No.6, 601-614

- [8] Cui J.Z and Shan Y.J. The Two-Scale Analysis Algorithms for the Structure with Several Configurations of Small Periodicity. Computational Techniques for Materials. Composites and Composite Structures Edited by B.H.V. Topping, CIVIL-COMP PRESS, 2000