Domain decomposition with non-matching grids for coupling of FEM and natural BEM

Ju'e Yang[1] Qiya Hu[2] Dehao Yu [3]
Institute of Computational Mathematics and Scientific/Engineering Computing,
Academy of Mathematical and System Sciences, The Chinese Academy of Sciences,
Beijing 100080, China.

Abstract

In this paper, we introduce a domain decomposition method with non-matching grids for solving Dirichlet exterior boundary problems by coupling of finite element method (FEM) and natural boundary element method(BEM). We first derive the optimal energy error estimate of the nonconforming approximation generated by this method. Then we apply a Dirichlet-Neumann(D-N) alternating algorithm to solve the coupled discrete system. It will be shown that such iterative method possesses the optimal convergence. The numerical experiments will testify our theoretical results.

Key words. domain decomposition, non-matching grids, natural boundary reduction, multiplier space, error estimate, D-N alternating algorithm, convergence.

AMS(MOS) subject classification. 65N30, 65N55

1. Introduction

The coupling method of finite element method (FEM) and boundary element method (BEM) ([14, 15, 9]) was developed as a generalization of the standard finite element method to problems in unbounded domains with complicated geometry shapes. It keeps all advantages of the finite element in treating the complicated bounded domains as well as the boundary element in treating unbounded domains. The standard coupling procedure can be described as following: the domain is decomposed into two subdomains, one bounded subdomain in which the standard finite element method will be used and the other unbounded subdomain where the boundary element method is applied. Finally, the unbounded domain problem is reduced to a bounded domain problem.

It is well known that domain decomposition methods (DDMs) are important numerical techniques for solving partial differential equations. When a domain is divided into some subdomains and artificial boundaries which called 'interface', the underlying partial differential equations can be solved independently in each subdomain. To get proper solution, the appropriate boundary condition must be given on the interface of sub-domains. When an unbounded domain is divided into some subdomains, there is at least one unbounded domain. In this case, the boundary reduction will be needed (see [8, 16, 18]).

[1] jueyang@lsec.cc.ac.cn.
[2] The work of this author was supported by Natural Science Foundation of China G10371129. (hqy@lsec.cc.ac.cn).
[3] The work of this author was supported by the Special Funds of State Major Basic Research Project G19990328 and the Knowledge Innovation Program of the Chinese Academy of Sciences (ydh@lsec.cc.ac.cn).

There are many ways to accomplish boundary reduction for unbounded domain problems. the natural boundary reduction proposed by Feng and Yu [3] has some distinctive advantages over the usual boundary reduction methods: the preservation of the symmetry and coerciveness, simplification of the discrete problem and the preservation of the optimal estimates and the numerical stability.

The existing coupling methods of FEM and natural BEM or the domain decomposition methods based on the natural boundary element reduction requires that the approximate solutions exactly satisfy the matching conditions over the interface or on the artificial boundary. This leads to some restrictions for the finite element discretizations on subdomain. Especially in the case of singularities of the solution, which has strong singularity near the concave vertex (see [8]). Therefore, we cannot expect that the approximate solution has an $O(h)$ estimation for the discretization error. In order to obtain the approximation of the solution which possesses satisfactory accuracy, it is necessary to use high refinements of the finite element grids near the concave vertices (see [8]). The DDMs with nonmatching grids can couple different variational problems in different sub-domains, see, for example, [1, 2, 4, 5, 6, 7, 11]. One important character of this method is to introduce a Lagrange multiplier space on the interface such that the matching conditions across the interface is replaced by a weaker one, i.e. the pointwise matching is replaced by the integral matching. Most importantly, the relaxation of the matching conditions on the interface still yields optimal approximation. Recently, there are increasingly interests in the field of domain decomposition methods with non-matching grids, however, there are little literature to study the case of unbounded domains.

In the present paper, we try to extend the DDMs with nonmatching grids to the problems in unbounded domains by the coupling of FEM and natural BEM. For simplicity of exposition, we consider only the case with two dimensions in the paper. In our method, the multiplier space is spanned by the dual basis multipliers presented in [11]. Such choice of the multiplier space can avoid the computation of the L^2 projector on the interface. In some particular (not general) situations, one can also choose the multiplier space consisting of piecewise constants on the elements (refer to [6]). We first derive an optimal error estimate of the resulting nonconforming approximation. Then, we design a Dirichlet-Neumann alternating algorithm to solve the coupled system. It will be shown that the iteration possesses a convergence rate independent of the mesh sizes. Our method can be extended to the case with three dimensions directly by a similar multiplier space (refer to [10]).

Our paper is organized as following. In section 2, we derive the coupled variational formulation for the Dirichlet exterior problem. In section 3 we make a discretization for the resulting coupled system based on the non-matching grids. In section 4, we give the error estimate for the approximates and obtain the optimal accuracy. In section 5, we apply the Dirichlet-Nuemann alternating method for solving this boundary value problem and prove the convergence of this iterative method. Finally, the numerical experiments testify the theoretical results.

2. Coupling of FEM and natural BEM

Let Ω is a bounded polygon domain in $\mathbb{R}^2$ and set $\Omega_e = \mathbb{R}^2 \backslash (\Omega \bigcup \partial\Omega)$ is the exterior domain of Ω. Consider the following Poisson equation for the Dirichlet exterior boundary problem:

$$\begin{cases} -\Delta u &= f \quad \text{in} \quad \Omega_{\rm e} \\ \quad u &= g \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.1}$$

where $f \in L^2(\Omega_e)$, and $g \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions, and u satisfies the asympotic conditions at the infinity:

$$u\left(x,y\right) = \beta + O\left(\tfrac{1}{r}\right) \qquad \text{as } r = \sqrt{x^2+y^2} \to \infty, \tag{2.2}$$

$$\left|\nabla u\left(x,y\right)\right| = O\left(\tfrac{1}{r^2}\right) \qquad \text{as } r = \sqrt{x^2+y^2} \to \infty, \tag{2.3}$$

where β is a constant.

Define the Sobolev space

$$W^1(\Omega_e) = \left\{ v \left| \frac{v}{\sqrt{1+r^2}\ln(2+r^2)}\,,\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y} \in L^2(\Omega_e) \right.\right\}, \tag{2.4}$$

and let

$$W_g^1(\Omega_e) \;=\; \left\{ v \left| \frac{v}{\sqrt{1+r^2}\ln(2+r^2)}\,,\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y} \in L^2(\Omega_e),\ v|_{\partial\Omega} = g \right.\right\}, \tag{2.5}$$

$$W_0^1(\Omega_e) \;=\; \left\{ v | v \in W_0^1(\Omega_e),\ v|_{\partial\Omega} = 0 \right\}. \tag{2.6}$$

The boundary value problem (2.1) is equivalent to the following variational form

$$\begin{cases} \text{find } u \in W_g^1(\Omega_e) \text{ such that} \\ D(u,v) = (f,v),\ \forall\ v \in W_0^1(\Omega_e) \end{cases} \tag{2.7}$$

where $D(u,v) = \int_{\Omega_e} \nabla u \cdot \nabla v dxdy$, and $(f,v) = \int_{\Omega_e} f v dxdy$.

Draw a circle Γ with radius R containing Ω. Γ divides Ω_e into two parts Ω_1 and Ω_2, where Ω_1 is the bounded domain between $\partial\Omega$ and Γ, and Ω_2 the unbounded domain outside Γ (see Figure 1).

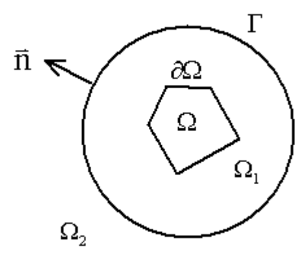

Figure 1: $\Omega_e = \Omega_1 \bigcup \Omega_2$ and $\Gamma = \Omega_1 \bigcap \Omega_2$ is an auxiliary boundary

To derive a coupling variational form equivalent to (2.7), we first define

$$H_g^1(\Omega_1) = \left\{ v | v \in H^1(\Omega_1), v|_{\partial\Omega} = g \right\}, \qquad H_0^1(\Omega_1) = \left\{ v | v \in H^1(\Omega_1), v|_{\partial\Omega} = 0 \right\}, \tag{2.8}$$

respectively, where

$$H^1(\Omega_1) = \left\{ v : v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega_1) \right\}, \tag{2.9}$$

Apply the natural boundary reduction to the exterior circular domain Ω_2, and the finite element method to the remaining bounded subdomian Ω_1. It is shown ([12, 13]) that the natural integral equation on Γ is

$$\frac{\partial u}{\partial \vec{n}} = -\frac{1}{4\pi R \sin^2 \frac{\theta}{2}} * u(R,\theta) = \mathcal{K}_2 u. \tag{2.10}$$

where $\vec{n}$ is the outward normal direction to Ω_2 and $\mathcal{K}_2$ is the natural integral operator with respect to the exterior domain Ω_2: $\mathcal{K}_2 : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$. Then using Green formula yields

$$\begin{aligned} D(u,v) &= \int_{\Omega_e} \nabla u \cdot \nabla v dxdy \\ &= \int_{\Omega_1} \nabla u \cdot \nabla v dxdy + \int_{\Omega_2} \nabla u \cdot \nabla v dxdy \\ &= \int_{\Omega_1} \nabla u \cdot \nabla v dxdy + \int_{\Gamma} \mathcal{K}_2 u \cdot v ds + \int_{\Gamma} v \cdot G_f ds \end{aligned} \tag{2.11}$$

with

$$G_f(p) = \int_{\Omega_2} \frac{\partial}{\partial \vec{n}} G(p,p') f(p')\, dp', \ \ p \in \Gamma, p' \in \Omega_2, \tag{2.12}$$

Here, $G(p,p')$ is the Green's function for the Laplace equation on the domain Ω_2.

Then we obtain the coupled FEM-BEM variational problem:

$$\begin{cases} \text{find } u \in H_g^1(\Omega_1), \text{ such that} \\ D_1(u,v) + D_2(u,v) = (f,v)_{\Omega_1} - \langle G_f, v \rangle_\Gamma, \quad \forall\, v \in H_0^1(\Omega_1) \end{cases} \tag{2.13}$$

where

$$D_1(u,v) = \int_{\Omega_1} \nabla u \cdot \nabla v dxdy, \qquad D_2(u,v) = \int_\Gamma v \mathcal{K}_2 u ds \tag{2.14}$$

are bilinear forms defined in Ω_1 and on Γ, respectively, and $\langle \cdot, \cdot \rangle_\Gamma$ denotes the L^2 inner product on Γ.

The operator $\mathcal{K}_2$ is just the Dirichlet-Neumann operator (Steklov-Poincaré operator) for Ω_2 ([17]). It is symmetric and semi-positive definite with respect to the inner product $\langle \cdot, \cdot \rangle_\Gamma$, (see[12, 13]).

Lemma 2.1 *The variational problem (2.13) has a unique solution $u \in H_g^1(\Omega_1)$.*

3. Non-matching grids and discretization

We will discrete the variational problem (2.13) independently in Ω_1 and on Γ, respectively. This leads to non-matching grids at the interface(compared [14, 16]). In the opposition to the original method the resulting discrete problem is never conforming because of the relaxation of the continuity condition at the interface. We'll show this variant preserves optimal energy error estimate in the next section.

We make a finite element triangulation $\mathcal{T}_{h_1}$(e.g. some regular quasi-uniform triangles and curved triangles at the interface Γ) in Ω_1 and let the total number of its nodes on Γ be N_1. Next we divide the auxiliary boundary Γ into N_2 equal segment arcs, which generate a division $\mathcal{T}_{h_2}$ on Γ, where $N_1 \geq N_2$. Here we note that the meshes may not match at the interface between the two subdomains, which means that the N_1 finite element nodes on Γ don't coincide with the N_2 boundary points on Γ. Therefore, the continuity conditions on the interface between Ω_1 and Ω_2 are broken, which are required for the usual coupling of FEM and natural BEM. It was pointed out in [1, 2, 11] that some weaker continuity condition across the interface can guarantee the optimal error estimate provided that the solution u is enough smooth. Here, our interface is a circle and our problem is posed on unbounded domain.

Let $V_{h_1}(\Omega_1) \subset H^1(\Omega_1)$ be the piecewise linear finite element space on Ω_1 with respect to $\mathcal{T}_{h_1}$ and $V_{h_2}(\Gamma)$ be the linear function space associated with $\mathcal{T}_{h_2}$. The parameter h is set equal to the 2-tuple(h_1, h_2) and the Lagrange multipliers space defined on Γ is denoted by $W_{h_2}(\Gamma)$ which will be discussed at detail later.

Define

$$\begin{aligned} V_h &= \left\{ v_h = (v_{h_1}, \psi_{h_2}) \in V_{h_1}(\Omega_1) \times V_{h_2}(\Gamma) : \langle v_{h_1}|_\Gamma - \psi_{h_2}, \mu \rangle_\Gamma = 0, \ \forall\ \mu \in W_{h_2}(\Gamma) \right\}, \\ V_h^0 &= \left\{ v_h \in V_h : v_{h_1}|_{\partial\Omega} = 0, \langle v_{h_1}|_\Gamma - \psi_{h_2}, \mu \rangle_\Gamma = 0, \ \forall\ \mu \in W_{h_2}(\Gamma) \right\}, \end{aligned} \tag{3.1}$$

Then we obtain the following non-conforming variational problem associated with (2.13),

$$\begin{cases} \text{find } (u_{h_1}, \varphi_{h_2}) \in V_h \text{ such that} \\ D_1(u_{h_1}, v_{h_1}) + D_2(\varphi_{h_2}, \psi_{h_2}) = (f, v_{h_1})_{\Omega_1} - \langle G_f, v_{h_1} \rangle_\Gamma, \quad \forall\ (v_{h_1}, \psi_{h_2}) \in V_h^0. \end{cases} \tag{3.2}$$

Remark 3.1 *The setting of Lagrange multipliers space would guarantee uniform ellipticity of this discrete problem(see [2]). Then it can be shown that the coupled variational problem (3.2) has unique solution* $(u_{h_1}, \varphi_{h_2}) \in V_h$.

As we have seen, the definition of Lagrange multipliers space is of great important for the unique solvability. We note that in [11] Barbara etc. used the dual basis functions to define the discrete Lagrange multiplier space for bounded domain problems in $\mathbb{R}^2$. Here we'll use the same dual basis to define a new type of multiplier space for unbounded domain problems.

Let γ be an any segmental arc on Γ and $\{l_i^\gamma\}_{i=1}^2$, $l_i^\gamma \in P_1(\gamma)$ be a linear basis on the element γ. Using linear Lagrange interpolation it is easily to know

$$l_1^\gamma(\theta) = \frac{N_2}{2\pi}(\theta_i - \theta), \qquad l_2^\gamma(\theta) = \frac{N_2}{2\pi}(\theta - \theta_{i-1}). \tag{3.3}$$

Define the test functions $\{\phi_i^\gamma\}_{i=1}^2$ satisfying

$$\left\langle l_i^\gamma, \phi_j^\gamma \right\rangle_\gamma = \delta_{ij} \left\langle l_i^\gamma, 1 \right\rangle_\gamma, \quad 1 \leq i, j \leq 2 \tag{3.4}$$

where δ_{ij} is the Kronecker delta symbol. Furthermore, we have

$$\left\langle l_i^\gamma, \left(\sum_{j=1}^{2} \phi_j^\gamma - 1 \right) \right\rangle_\gamma = 0, \quad 1 \leq i \leq 2. \tag{3.5}$$

Therefore, from (3.3) (3.4) we deduce

$$\phi_1^\gamma(\theta) = 2l_1^\gamma(\theta) - l_2^\gamma(\theta) = \frac{N_2}{2\pi}(2\theta_i + \theta_{i-1} - 3\theta), \tag{3.6}$$

$$\phi_2^\gamma(\theta) = -l_1^\gamma(\theta) + 2l_2^\gamma(\theta) = \frac{N_2}{2\pi}(3\theta - \theta_i - 2\theta_{i-1}). \tag{3.7}$$

Let $\{L_i(\theta)\}_{i=1}^{N_2}$ and $\{\Phi_i(\theta)\}_{i=1}^{N_2}$ be two global nodal basis functions for Γ (see Figure 2 and Figure 3). As a consequence, we set $W_{h_2}(\Gamma) = \text{span}\{\Phi_i(\theta),\ 1 \le i \le N_2\}$. Under uniform subdivision the piecewise linear basis functions are

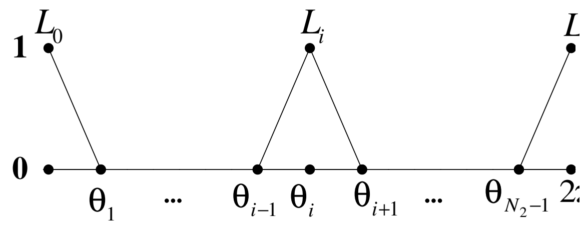

Figure 2: Basis functions of $V_{h_2}(\Gamma)$

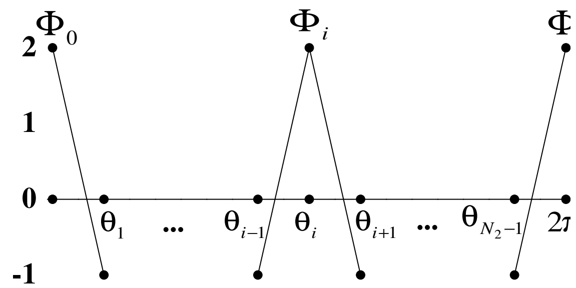

Figure 3: Dual basis functions of $W_{h_2}(\Gamma)$ with a circle interface Γ

$$\Phi_i(\theta) = \begin{cases} \frac{N_2}{2\pi}(3\theta - \theta_i - 2\theta_{i-1}) & \theta_{i-1} \le \theta \le \theta_i \\ \frac{N_2}{2\pi}(2\theta_{i+1} + \theta_i - 3\theta) & \theta_i \le \theta \le \theta_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

where $i = 1,\ 2,\ \cdots,\ N_2$ and $\theta_i = \frac{2\pi}{N_2} i$.

Since $V_{h_2}(\Gamma) \subset H^{1/2}(\Gamma)$, the test functions space $W_{h_2}(\Gamma)$ may be embedded in the dual space of $H^{1/2}(\Gamma)$ with respect to the L^2-inner product. Therefore, we obtain $W_{h_2}(\Gamma) \subset H^{-1/2}(\Gamma)$. $\{\Phi_i(\theta)\}_{i=1}^{N_2}$ is called as the dual basis defined on Γ. From the observation of Figure 3, for any fixed node θ_i on Γ, the dual basis function $\Phi_i(\theta)$ has its support on two mesh intervals and it increases linearly from minimum value -1 to the maximum value 2 on the first interval and decreases linearly from 2 to -1 on the second interval.

Similar to (3.4), $L_i(\theta)$ and $\Phi_i(\theta)$ also satisfy the following global property:

$$\langle L_i(\theta),\ \Phi_j(\theta)\rangle_\Gamma = \delta_{ij} \langle L_i(\theta),\ 1\rangle_\Gamma, \quad 1 \le i, j \le N_2. \tag{3.9}$$

Remark 3.2 *The construction of the multipliers basis functions here is somewhat different. Figure 3 and Figure 4 illustrate the major difference. In the earlier papers about the dual* *multipliers approach(see [11]), the multipliers basis space corresponds to only the interior nodes* $a_2, \cdots, a_{n-1}$ *of the interface* Γ *and they are constants in the intervals* $[a_1, a_2]$ *and* $[a_{n-1}, a_n]$ *(identically 1); See Figure 4. Note that they are discontinuous across the corners and the dimension of multipliers space equals the total number of interior nodes in the interface. But in the case of our model problem, since the interface of* Ω_1 *and* Ω_2 *is a circle, there doesn't exist any corner points between two sub-domains. So the dual multipliers basis functions that we obtained on* Γ *are piecewise linear continuous (see Figure 3) and the dimension of* $W_{h_2}(\Gamma)$ *equals* N_2*.*

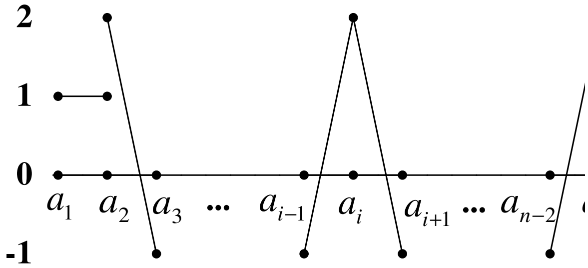

Figure 4: Multipliers basis functions with corner points

Next, in order to obtain some optimal error estimation we introduce some projection operations which possess good properties. Let $P_{h_2} : L^2(\Gamma) \to W_{h_2}(\Gamma)$ be the L^2 orthogonal projection operator. The following approximation result of this operator can be verified in the standard manner.

Lemma 3.1 *For any function* ψ *of* $H^s(\Gamma)$, $0 \le s \le 1$, *there exists a constant c such that*

$$||\psi - P_{h_2}\psi||_{0,\Gamma} \le ch^s|\psi|_{s,\Gamma}. \tag{3.10}$$

As in [**?**, 11] a mortar projection play the central role to analyze the approximation error. Here we define a projection operator $\Pi_{h_2} : L^2(\Gamma) \to V_{h_2}(\Gamma)$. Given a $\psi \in L_2(\Gamma)$, $\Pi_{h_2}\psi \in V_{h_2}(\Gamma)$ satisfies

$$\int_\Gamma (\psi - \Pi_{h_2}\psi)\mu ds = 0 \quad \forall\ \mu \in W_{h_2}(\Gamma). \tag{3.11}$$

We'll need the stability property of this operator which is given in the following lemma.

Lemma 3.2 *The operator* Π_{h_2} *is stable in* $H^{1/2}(\Gamma)$*, i.e. there is a constant c such that*

$$||\Pi_{h_2}\psi||_{\frac{1}{2},\Gamma} \le c||\psi||_{\frac{1}{2},\Gamma} \qquad \forall\psi \in H^{1/2}(\Gamma). \tag{3.12}$$

Proof: From the definition of the operator Π_{h_2}, $\forall\psi \in L^2(\Gamma), \Pi_{h_2}\psi \in V_{h_2}(\Gamma)$, it can be written

$$\Pi_{h_2}\psi = \sum_{i=1}^{N_2} \alpha_i L_i \tag{3.13}$$

Using (3.11) and the global orthogonality relation (3.9) with direct calculation, we have

$$\begin{aligned}
||\Pi_{h_2}\psi||^2_{0,\Gamma} &= \left(\sum_{i=1}^{N_2} \frac{\int_\Gamma \psi\Phi_i ds}{\int_\Gamma L_i ds} L_i\right)^2 \\
&\le \sum_{i=1}^{N_2} \frac{\int_\tau L_i^2 ds \int_\tau \Phi_i^2 ds}{\left(\int_\tau \text{L}_i ds\right)^2} ||\psi||^2_{0,\text{supp }(L_i)} \le c||\psi||^2_{0,\Gamma} \qquad (3.14)
\end{aligned}$$

in which τ denotes the support of $L_i(\theta)$ and we used the Hölder inequality.

Note that the constants are contained in the space $V_{h_2}(\Gamma)$. For a nodal point $a \in \mathcal{T}_{h_2}$, let γ is the segmental arc with a as one of its endpoints. Set $\overline{\psi}(a)$ as the average of ψ over γ, i.e.

$$\overline{\psi}(a) = \frac{1}{\text{meas}(\gamma)} \int_\gamma \psi ds \qquad (3.15)$$

The stability of Π_{h_2} over $H^1(\Gamma)$ can be obtained by using an inverse inequality, from its L^2-stability

$$\begin{aligned}
|\Pi_{h_2}\psi|_{1,\gamma} &= |\Pi_{h_2}\left(\psi - \overline{\psi}(a)\right)|_{1,\gamma} \le \frac{c}{h_2}||\Pi_{h_2}\left(\psi - \overline{\psi}(a)\right)||_{0,\gamma} \\
&\le \frac{c}{h_2}||\psi - \overline{\psi}(a)||_{0,\gamma} \le c|\psi|_{1,\gamma}. \qquad (3.16)
\end{aligned}$$

In the last inequality we apply the Friedrichs' inequality. Then by summing these estimation over all segmental arcs we obtain the H^1-stability of Π_{h_2} on Γ.

Finally, the stability of Π_{h_2} over $H^{\frac{1}{2}}(\Gamma)$ is easily derived from (3.14) and (3.16) by interpolation argument.

4. Analysis of error

An crucial point concerned with the non-matching grids which lead to nonconformity is that the solution u_h approximates the exact u with the same accuracy as a corresponding conforming finite element solution.

Define the norm

$$||v_h|| = \left(||v_{h_1}||^2_{1,\Omega_1} + ||\psi_{h_2}||^2_{\frac{1}{2},\Gamma}\right)^{1/2}, \quad \forall\ v_h \in V_h. \qquad (4.1)$$

Lemma 4.1 *From the second Strang's lemma, we have*

$$||u - u_h|| \le \inf_{\forall v_h \in V_h} ||u - v_h|| + \sup_{\forall v_h \in V_h} \frac{\int_\Gamma \frac{\partial u}{\partial \vec{n}}[v_h] ds}{||v_h||}, \qquad (4.2)$$

where $[v_h]$ *denotes the jump of this function through* Γ.

We note that the first term of 4.2 is the best approximation error, while the second term is the consistency error. The best approximation error can be estimated by using interpolation inequalities for conforming finite elements and stability properties of the projection Π_{h_2}; For estimation of the consistency error, we use the fact the jump of the solution is orthogonal to the multiplier space $W_{h_2}(\Gamma)$. Thus we have the following theorem.

Theorem 4.1 *Assume that* $u|_{\Omega_1} \in H^{1+\alpha}(\Omega_1)$, $\frac{1}{2} < \alpha < 1$ *and* $u|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$, *then there exists a constant* c *independent of the mesh parameters* h_1 *and* h_2 *such that the following error estimate holds,*

$$||u - u_h|| \leq c(h_1^\alpha ||u||_{1+\alpha,\Omega_1} + h_2 ||u||_{\frac{3}{2},\Gamma}). \tag{4.3}$$

Proof: Step 1: we estimate the best approximation error.

Let $\pi_{h_i}, i = 1,2$ are the Lagrange interpolation operators in Ω_1 and on Γ, respectively. Then we define v_h by

$$v_{h_1} = \pi_{h_1} u, \qquad \psi_{h_2} = \pi_{h_2}(u|_\Gamma) + \Pi_{h_2}[\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)]. \tag{4.4}$$

By this way, we have

$$\langle v_{h_1}|_\Gamma - \psi_{h_2}, \mu\rangle_\Gamma = \langle [\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)] - \Pi_{h_2}[\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)], \mu\rangle_\Gamma = 0, \tag{4.5}$$

and so v_h satisfies the weak continuity condition in the interface, then it belongs to V_h.

Applying the trace theorem and the $H^{\frac{1}{2}}$-stability of Π_{h_2}, we have

$$\begin{aligned} & \inf_{\forall v_h \in V_h} ||u - v_h|| \leq \inf_{\forall v_h \in V_h} \left(||u|_{\Omega_1} - v_{h_1}||_{1,\Omega_1} + ||u|_\Gamma - \psi_{h_2}||_{\frac{1}{2},\Gamma} \right) \\ \leq \; & ||u - \pi_{h_1} u||_{1,\Omega_1} + ||u - \pi_{h_2} u - \Pi_{h_2}[\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)]||_{\frac{1}{2},\Gamma} \\ \leq \; & ||u - \pi_{h_1} u||_{1,\Omega_1} + ||u - \pi_{h_2} u||_{\frac{1}{2},\Gamma} + ||\Pi_{h_2}[\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)]||_{\frac{1}{2},\Gamma} \\ \leq \; & c h_1^\alpha ||u||_{1+\alpha,\Omega_1} + c h_2 ||u||_{3/2,\Gamma} + c||\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)||_{\frac{1}{2},\Gamma.} \end{aligned} \tag{4.6}$$

We use the triangle inequality and get

$$\begin{aligned} ||\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)||_{\frac{1}{2},\Gamma} &\leq ||\pi_{h_1} u|_\Gamma - \pi_{h_2}(u|_\Gamma)||_{\frac{1}{2},\Gamma} \\ &\leq ||u - \pi_{h_1} u||_{\frac{1}{2},\Gamma} + ||u - \pi_{h_2}(u|_\Gamma)||_{\frac{1}{2},\Gamma} \\ &\leq c h_1^\alpha ||u||_{1+\alpha,\Omega_1} + c h_2 ||u||_{\frac{3}{2},\Gamma.} \end{aligned} \tag{4.7}$$

Step 2: we estimate the consistency error

$$\begin{aligned} \left| \int_\Gamma \frac{\partial u}{\partial \vec{n}} [v_h] ds \right| &= \left| \int_\Gamma \frac{\partial u}{\partial \vec{n}} (v_{h_1}|_\Gamma - \psi_{h_2}) ds \right| \\ &= \left| \int_\Gamma \left(\frac{\partial u}{\partial \vec{n}} - P_{h_2}\left(\frac{\partial u}{\partial \vec{n}}\right)\right) (v_{h_1}|_\Gamma - \psi_{h_2})\, ds \right|. \end{aligned} \tag{4.8}$$

From the property of the jump in the interface and the definition of operator P_{h_2}, we know

$$P_{h_2}(v_{h_1}|_\Gamma) = P_{h_2}\psi_{h_2}. \tag{4.9}$$

Using the Cauchy-Schwarz inequality and Lemma 3.1, we have

$$\begin{aligned}
\left|\int_\Gamma \frac{\partial u}{\partial \vec{n}}[v_h]ds\right| &\le c\left|\left|\frac{\partial u}{\partial \vec{n}} - P_{h_2}\frac{\partial u}{\partial \vec{n}}\right|\right|_{0,\Gamma} ||v_{h_1}|_\Gamma - \psi_{h_2}||_{0,\Gamma} \\
&\le ch_2^{1/2}\left|\frac{\partial u}{\partial \vec{n}}\right|_{\frac{1}{2},\Gamma}(||v_{h_1}|_\Gamma - P_{h_2}(v_{h_1}|_\Gamma) + P_{h_2}(v_{h_1}|_\Gamma) - \psi_{h_2}||_{0,\Gamma}) \\
&= ch_2^{1/2}\left|\frac{\partial u}{\partial \vec{n}}\right|_{\frac{1}{2},\Gamma}(||v_{h_1}|_\Gamma - P_{h_2}(v_{h_1}|_\Gamma) + P_{h_2}\psi_{h_2} - \psi_{h_2}||_{0,\Gamma}) \\
&\le ch_2^{1/2}\left|\frac{\partial u}{\partial \vec{n}}\right|_{\frac{1}{2},\Gamma}(||v_{h_1}|_\Gamma - P_{h_2}(v_{h_1}|_\Gamma)||_{0,\Gamma} + ||P_{h_2}\psi_{h_2} - \psi_{h_2}||_{0,\Gamma}||_{0,\Gamma}) \\
&\le ch_2^{1/2}\left|\frac{\partial u}{\partial \vec{n}}\right|_{\frac{1}{2},\Gamma}\left(h_2^{1/2}|v_{h_1}|_{\frac{1}{2},\Gamma} + h_2^{1/2}|\psi_{h_2}|_{\frac{1}{2},\Gamma}\right) \\
&\le ch_2||u||_{\frac{3}{2},\Gamma}||v_h||. \qquad (4.10)
\end{aligned}$$

In the last inequality we have used the trace theorem.

Remark 4.1 *In order to obtain the optimal error estimation, we should apply high refinements of finite element grids in Ω_1 such that the fine mesh size h_1 and the coarse mesh h_2 satisfy $h_1^\alpha \approx h_2$. In fact if α is small enough, it requires that $h_1 \ll h_2$. From the main theorem we can see that the dual basis mortar approach leads to a stable and optimally convergent approximation.*

5. D-N alternating method

In this section we design a Dirichlet-Neumann alternating iteration for solving the boundary value problem (2.1). For simplicity of exposition, we here discuss only the D-N alternating algorithm of continuous problems. With obvious modifications, we can study the case of discrete problems (refer to [18]).

1. Choose initial value $\lambda^0 \in H^{\frac{1}{2}}(\Gamma), n := 0$.
2. Solve the exterior Dirichlet problem on Ω_2:
$$\begin{cases} -\Delta u_2^n &= f \quad \text{in } \Omega_2 \\ u_2^n &= \Pi_{h_2}\lambda^n \quad \text{on } \Gamma \end{cases}. \qquad (5.1)$$
3. Solve the Neumann mixed boundary value problem in interior domain Ω_1:
$$\begin{cases} -\Delta u_1^n &= f \quad \text{in } \Omega_1 \\ \dfrac{\partial u_1^n}{\partial \vec{n}} &= \dfrac{\partial u_2^n}{\partial \vec{n}} \quad \text{on } \Gamma \\ u_1^n &= g \quad \text{on } \partial\Omega \end{cases}. \qquad (5.2)$$
4. Set $\lambda^{n+1} = \theta_n u_1^n + (1-\theta_n)\lambda^n$ on Γ.
5. Let $n := n+1$, then goto the second step.

Remark 5.1 *The nodal values of projection operator* Π_{h2} *on* Γ *are uniquely determined by its values in the opposite side of* Γ*. In fact, these values can be obtained by the direct calculation due to (3.9)and (3.11). In particular, for any function* $\lambda \in V_{h_1}(\Gamma)$, $\Pi_{h_2}\lambda \in V_{h_2}(\Gamma)$*. Let* $a_i, i = 1, 2, \cdots, N_2$ *be the nodal points on* Γ*, then,*

$$\left(\Pi_{h_2}\lambda|_\Gamma\right)(a_i) = \frac{\int_\Gamma \lambda \Phi_i ds}{\int_\Gamma L_i ds}, \quad i = 1, 2, \cdots, N_2. \tag{5.3}$$

Besides, in the discrete problems, the second equation of (5.2) holds in the sense of integral, i.e.,

$$\int_\Gamma \frac{\partial u_1^n}{\partial \vec{n}} v ds = \int_\Gamma \frac{\partial u_2^n}{\partial \vec{n}} v ds, \ \forall v \in V_{h_1(\Omega_1)}. \tag{5.4}$$

Observing that only the approximation of the normal derivative of u_2^n on the interface Γ is required for solving the mixed boundary value problem in the bounded sub-domain Ω_1, it make sense to apply the Poisson integral formula based on the theory of natural boundary integral method [13] since the equation (5.1) is a Dirichlet boundary value problem in the unbounded circular exterior sub-domain Ω_2.

$$u_2^n(r,\theta) = \frac{r^2 - R^2}{2\pi} \int_0^{2\pi} \frac{\lambda^n(\theta')}{R^2 + r^2 - 2Rr\cos(\theta - \theta')} d\theta' + \int\int_{\Omega_2} G(p,p')^{(2)} f(p') dp' \tag{5.5}$$

and

$$\frac{\partial u_2^n}{\partial \vec{n}}(\theta) = -\frac{1}{4\pi R} \int_0^{2\pi} \frac{\lambda^n(\theta')}{\sin^2 \frac{\theta - \theta'}{2}} d\theta + \int\int_{\Omega_2} \left(\frac{\partial}{\partial \vec{n}} G^{(2)}(p,p')\right) f(p') dp', \tag{5.6}$$

where the Green function in infinite domain Ω_2 can be expressed by

$$G^{(2)}(p,p') = \frac{1}{4\pi} \ln \frac{R^4 + r^2 r'^2 - 2Rrr'\cos(\theta - \theta')}{R^2(r^2 + r'^2 - 2rr'\cos(\theta - \theta'))} \tag{5.7}$$

and $p = (r, \theta)$, $p' = (r', \theta')$ are the polar coordinates,respectively.

Thus, it is not necessary to actually solve the equation (5.1). Therefore the D-N alternating algorithm can greatly reduced the computational work. We only treat the mixed boundary value problem in the rather small bounded domain where the standard finite element method can be used.

In order to analyze the convergence of the D-N alternating algorithm, define harmonic extensions operators R_1 and R_2 as follows:

$$R_1 : H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega_1)$$

which means that for any $\lambda \in H^{\frac{1}{2}}(\Gamma)$, $R_1\lambda \in H^1(\Omega_1)$ and satisfies

$$\begin{cases} -\Delta R_1 \lambda &= 0, & \text{in } \Omega_1 \\ R_1\lambda &= \lambda, & \text{on } \Gamma \\ R_1\lambda &= 0, & \text{on } \partial\Omega \end{cases}, \tag{5.8}$$

$$R_2 : H^{\frac{1}{2}}(\Gamma) \to H^*(\Omega_2) \tag{5.9}$$

where $H^*(\Omega_2)$ denote the space that the function in it satisfies the equation $-\Delta u = f,$ in Ω_2 and some suitable condition at the infinity. Then we have

$$\begin{cases} -\Delta R_2 \Pi_{h_2} \lambda &= 0, \qquad \text{on } \Omega_2 \\ R_2 \Pi_{h_2} \lambda &= \Pi_{h_2} \lambda, \qquad \text{on } \Gamma \end{cases} . \tag{5.10}$$

λ should satisfy the interface condition $\dfrac{\partial u_1(\lambda)}{\partial \vec{n}} = \dfrac{\partial u_2(\lambda)}{\partial \vec{n}}$ on the auxiliary boundary Γ. From (5.1) and (5.2), we have

$$\frac{\partial}{\partial \vec{n}}(R_2 \Pi_{h_2} - R_1)\lambda = \chi \tag{5.11}$$

where χ is the function independent of λ which can be solved beforehand in the sub-domains. Write $S = \dfrac{\partial}{\partial \vec{n}}(R_2 \Pi_{h_2} - R_1)$ as the Steklov-Poincaré operator on Γ. So we have $S\lambda = \chi$, where $S = S_1 + S_2,\ S_1 = -\dfrac{\partial}{\partial \vec{n}}(R_1 \cdot),\ S_2 = \dfrac{\partial}{\partial \vec{n}}(R_2 \Pi_{h_2} \cdot)$.

Lemma 5.1 *The D-N alternating method is equivalent to the associated preconditioned Richardson iterative method:*

$$S_1\left(\lambda^{n+1} - \lambda^n\right) = \theta_n \left(\chi - S\lambda^n\right). \tag{5.12}$$

Proof: Let $e_k^n = u - u_k^n,\ k = 1, 2$ and $\mu^n = \lambda - \Pi_{h_2}\lambda^n$, where $\lambda = u|_\Gamma$. Then the error terms e_1^n and e_2^n satisfy the following equations, respectively,

$$\begin{cases} -\Delta e_2^n &= 0, \qquad \text{in } \Omega_2 \\ e_2^n &= \mu^n, \qquad \text{on } \Gamma \end{cases} , \tag{5.13a}$$

$$\begin{cases} -\Delta e_1^n &= 0, \qquad \text{in } \Omega_1 \\ e_1^n &= 0, \qquad \text{on } \partial\Omega \\ \dfrac{\partial e_1^n}{\partial \vec{n}} &= \dfrac{\partial e_2^n}{\partial \vec{n}}, \qquad \text{on } \Gamma \end{cases} , \tag{5.13b}$$

and

$$\mu^{n+1} = \theta_n \Pi_{h_2}(e_1^n|_\Gamma) + (1 - \theta_n)\mu^n. \tag{5.14}$$

Therefore,

$$e_1^n = R_1\left(e_1^n|_\Gamma\right), \qquad e_2^n = R_2\left(e_2^n|_\Gamma\right) = R_2\mu^n, \tag{5.15}$$

we have

$$S_1(e_1^n|_\Gamma) = -\frac{\partial}{\partial \vec{n}}(R_1 e_1^n|_\Gamma) = -\frac{\partial}{\partial \vec{n}}(e_2^n|_\Gamma) = -\frac{\partial}{\partial \vec{n}}(\lambda - \Pi_{h_2}\lambda^n) = -S_2(\lambda - \lambda^n). \tag{5.16}$$

Since

$$\lambda^{n+1} - \lambda^n = \theta_n(u_1^n|_\Gamma - \lambda^n), \tag{5.17}$$

therefore,

$$\begin{aligned} S_1(\lambda^{n+1} - \lambda^n) &= S_1[\theta_n(u_1^n|_\Gamma - \lambda^n)] = \theta_n[S_1(u_1^n|_\Gamma - \lambda) + S_1(\lambda - \lambda^n)] \\ &= \theta_n(S_1 + S_2)(\lambda - \lambda^n) = \theta_n S(\lambda - \lambda^n) = \theta_n\left(\chi - S\lambda^n\right). \end{aligned} \tag{5.18}$$

The proof is completed.

Theorem 5.1 *The D-N alternating algorithm is convergent.*

Proof: Applying Green formula in the bounded domain Ω_1, we have,

$$D_1(R_1\lambda, R_1\lambda) = \int_\Gamma \frac{\partial R_1\lambda}{\partial \vec{n}} R_1\lambda ds. \tag{5.19}$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product on Γ, then,

$$\langle S_1\lambda, \lambda\rangle = D_1(R_1\lambda, R_1\lambda), \quad \langle S_2\lambda, \lambda\rangle = D_2(R_2\Pi_{h_2}\lambda, R_2\Pi_{h_2}\lambda). \tag{5.20}$$

According to the stability property (3.12) of project operator Π_{h_2} we can deduce that,

$$|D_2(R_2\Pi_{h_2}\lambda, R_2\Pi_{h_2}\lambda)| \leq c||\Pi_{h_2}\lambda||^2_{\frac{1}{2},\Gamma} \leq c||\lambda||^2_{\frac{1}{2},\Gamma} \tag{5.21}$$

Then we use the trace theorem and obtain

$$||\lambda||^2_{\frac{1}{2},\Gamma} \leq c||R_1\lambda||^2_{1,\Omega_1} \leq cD_1(R_1\lambda, R_1\lambda). \tag{5.22}$$

In the above discussion, the constant c is independent of mesh parameter. So,

$$1 \leq \left|\frac{\langle S\lambda, \lambda\rangle}{\langle S_1\lambda, \lambda\rangle}\right| = \left|1 + \frac{D_2(\Pi_{h_2}\lambda, \Pi_{h_2}\lambda)}{D_1(\lambda, \lambda)}\right| \leq 2, \tag{5.23}$$

which means the spectral equivalence of S and S_1. Thus the condition number of $S_1^- S$ is independent of the parameter of the mesh h_2. So the Richardson iteration is convergent as well as the D-N alternating algorithm from Lemma 5.1.

6. Numerical examples

In this section, we give some numerical results to illustrate the theoretical results obtained in the paper. For numerical testing we consider the exterior Dirichlet problem of Ω

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega_{\mathrm{e}} \\ \quad u &= g \quad \text{on } \partial\Omega \end{cases} \tag{6.1}$$

where Ω is a square whose side length is $2c$ (c is some constant) in $\mathbb{R}^2$ and $\Omega_e = \mathbb{R}^2 \backslash (\Omega \bigcup \partial\Omega)$. Draw a circumference Γ with radius R enclosing $\partial\Omega$, then Ω_e is divided into an interior subdomain Ω_1 and an exterior subdomain Ω_2 by the artificial boundary Γ. Here, $f = 0$ and the boundary condition g on $\partial\Omega$ is computed from a given harmonic function u, i.e. $g = u(x, y)|_{\partial\Omega}$ such that the exact solution is $u(x, y) = \frac{x}{x^2+y^2}$.

We divide Γ into N_2 equal segmental arcs and make a triangulation in Ω_1 whose number of elements is NE, associated with N_1 nodes on Γ. In this test we use piecewise linear finite element in Ω_1, $N_1 = 2N_2$ and take $\theta_n = \theta = 0.5$ and initial guess $\lambda^0 = 0.0$. The numerical solution u_h is compared with the true solution u with respect to L^2-error and the maximum node error, respectively, in the following tables.

Table 1: The maximum node error when R=2.0 and $c = 1$

				steps	of	iteration				
N_1	NE	0	1	2	3	4	5	6	7	8
16	64	0.6666	0.2857	0.0795	0.0354	0.0244	0.0215	0.0207	0.0204	0.0204
32	256	0.8000	0.2655	0.0626	0.0197	0.0088	0.0059	0.0051	0.0048	0.0048
64	1024	0.8888	0.2614	0.0590	0.0160	0.0065	0.0022	0.0021	0.0012	0.0019

Table 2: The L^2-error when R=2.0 and $c = 1$

				steps	of	iteration				
N_1	NE	0	1	2	3	4	5	6	7	8
16	64	1.0336	0.3167	0.0956	0.0437	0.0327	0.0303	0.0297	0.0296	0.0295
32	256	1.2007	0.3137	0.0828	0.0253	0.0112	0.0084	0.0078	0.0076	0.0076
64	1024	1.2790	0.3156	0.0805	0.0218	0.0050	0.0028	0.0014	0.0019	0.0011

Table 3: The Maximum node error when R=4.0 and c=1

			steps	of	iteration				
N_1	NE	0	1	2	3	4	5	6	7
16	64	0.4000	0.2638	0.0569	0.0382	0.0371	0.0370	0.0370	0.0370
32	256	0.5714	0.2227	0.0268	0.0128	0.0124	0.0123	0.0123	0.0123
64	1024	0.7272	0.2129	0.0191	0.0042	0.0044	0.0045	0.0045	0.0045

Table 4: The L^2-error when R=4.0 and c=1

			steps	of	iteration				
N_1	NE	0	1	2	3	4	5	6	7
16	64	1.3938	0.8797	0.2690	0.2189	0.2143	0.2138	0.2137	0.2137
32	256	1.6931	0.7299	0.1085	0.0622	0.0589	0.0586	0.0585	0.0585
64	1024	1.8467	0.6948	0.0671	0.0182	0.0152	0.0150	0.0149	0.0149

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